

3.2 Proving Identities

In this section we will be studying techniques for verifying trigonometric identities. We need to show that each of these equations is true for all values of our variable. There is no well defined set of rules for how to verify an identity but we do have some guidelines we can use.

Guidelines for Verifying Trigonometric Identities

1. Only work with one side of the equation at a time. It is usually better to work with the more complicated side first.
2. Use algebraic techniques: Factor an expression, add fractions, expand an expression, or multiply by a conjugate to create a simpler expression.
3. Look for ways to use the fundamental identities from section 3.1. Pay attention to what is in the expression you want. Sines and cosines work well together, as do secants and tangents, as do cosecants and cotangents.
4. Convert everything to sines and cosines and then use the fundamental identities.
5. Always try something. Even paths that don't end up where you want may provide insight.

NOTE: When you verify an identity you cannot assume that both sides of the equation are equal because you are trying to verify that they are equal. This means that you cannot use operations that do the same thing to both sides of the equation such as multiplying the same quantity to both sides or cross multiplication.

Example 3.2.1

Verify the identity $\cos x + \sin x \tan x = \sec x$.

Solution: We will work with the left side of the equation, because it is more complicated, and make it look like the right side.

$$\begin{aligned}
 \cos x + \sin x \tan x &= \cos x + \sin x \left(\frac{\sin x}{\cos x} \right) && \text{identity (3.4)} \\
 &= \cos x \left(\frac{\cos x}{\cos x} \right) + \sin x \left(\frac{\sin x}{\cos x} \right) && \text{common denominator} \\
 &= \frac{\cos^2 x}{\cos x} + \frac{\sin^2 x}{\cos x} \\
 &= \frac{\cos^2 x + \sin^2 x}{\cos x} \\
 &= \frac{1}{\cos x} && \text{identity (3.5)} \\
 &= \sec x
 \end{aligned}$$

Example 3.2.2

Verify the identity $\frac{\sec x - 1}{1 - \cos x} = \sec x$.

Solution 1: The left side is certainly more complicated so we will start there. The fraction doesn't have any squared terms so we can't use the Pythagorean identities and there isn't any algebraic simplification that can be done. We will convert the secant to cosine and then simplify.

$$\begin{aligned} \frac{\sec x - 1}{1 - \cos x} &= \frac{\frac{1}{\cos x} - 1}{1 - \cos x} && \text{convert to cosine} \\ &= \frac{\left(\frac{1}{\cos x} - 1\right)(\cos x)}{(1 - \cos x)(\cos x)} && \text{multiply by } 1 = \frac{\cos x}{\cos x} \\ &= \frac{1 - \cos x}{(1 - \cos x)(\cos x)} && \text{simplify} \\ &= \frac{1}{\cos x} \\ &= \sec x \end{aligned}$$

Solution 2: We will show a different way to verify the identity. This method is longer but it illustrates that there is often more than one way to solve the problems. The fraction doesn't have any squared terms so we can't use the Pythagorean identities, however, we can multiply by the conjugate of the denominator to make it look like a Pythagorean identity. Remember that $(a + b)(a - b) = a^2 - b^2$ so here if we multiply $(1 - \cos x)(1 + \cos x) = 1 - \cos^2 x$. This technique is known as "multiplying by the conjugate." A conjugate is an expression where the sign has been changed. The conjugate of $a + b$ is $a - b$ and vice versa. We can't just multiply the denominator by something because that changes the problem. What we need to do is multiply by a clever form of 1. We will multiply by $1 = \frac{1 + \cos x}{1 + \cos x}$.

$$\begin{aligned} \frac{\sec x - 1}{1 - \cos x} &= \left(\frac{\sec x - 1}{1 - \cos x}\right) \left(\frac{1 + \cos x}{1 + \cos x}\right) && \text{multiply by } 1 \\ &= \frac{\sec x + (\sec x)(\cos x) - \cos x - 1}{1 - \cos^2 x} \\ &= \frac{\frac{1}{\cos x} + \left(\frac{1}{\cos x}\right)(\cos x) - \cos x - 1}{1 - \cos^2 x} && \text{reciprocal identity} \end{aligned}$$

$$\begin{aligned}
&= \frac{\frac{1}{\cos x} + 1 - \frac{\cos^2 x}{\cos x} - 1}{1 - \cos^2 x} && \text{Simplify and find common denominator} \\
&= \frac{1 - \cos^2 x}{1 - \cos^2 x} && \text{Simplify} \\
&= \frac{1 - \cos^2 x}{(\cos x)(1 - \cos^2 x)} && \text{Simplify} \\
&= \frac{1}{\cos x} && \text{Simplify} \\
&= \sec x
\end{aligned}$$

Example 3.2.3

Verify the identity $\frac{\sec x + \tan x}{\sec x - \tan x} = (\sec x + \tan x)^2$

Solution: Here we will work with the left side and multiply by the conjugate of the denominator. We need to multiply by $1 = \frac{\sec x + \tan x}{\sec x + \tan x}$ and then use identity (3.6).

$$\begin{aligned}
\frac{\sec x + \tan x}{\sec x - \tan x} &= \left(\frac{\sec x + \tan x}{\sec x - \tan x} \right) \left(\frac{\sec x + \tan x}{\sec x + \tan x} \right) && \text{multiply by 1} \\
&= \frac{(\sec x + \tan x)^2}{\sec^2 x - \tan^2 x} && \text{simplify} \\
&= \frac{(\sec x + \tan x)^2}{1} && \text{identity (3.6)} \\
&= (\sec x + \tan x)^2
\end{aligned}$$

Example 3.2.4

Verify the identity $\frac{\sin x \cos x}{\sin x - \cos x} = \cos x - \frac{\cos x}{1 - \tan x}$

Solution: Neither side of this problem looks simple but the right hand side involves two fractions. That is more complicated than the one on the left so we will begin there.

$$\cos x - \frac{\cos x}{1 - \tan x} = \cos x - \frac{\cos x}{1 - \frac{\sin x}{\cos x}} \quad \text{Convert to sine and cosine}$$

$$\begin{aligned}
&= \cos x - \frac{(\cos x)(\cos x)}{\left(1 - \frac{\sin x}{\cos x}\right)(\cos x)} && \text{Multiply by } 1 = \frac{\cos x}{\cos x} \\
&= \cos x - \frac{\cos^2 x}{\cos x - \sin x} && \text{Simplify} \\
&= \frac{(\cos x)(\cos x - \sin x)}{\cos x - \sin x} - \frac{\cos^2 x}{\cos x - \sin x} && \text{Find a common denominator} \\
&= \frac{\cos^2 x - \cos x \sin x - \cos^2 x}{\cos x - \sin x} && \text{Combine the fractions} \\
&= \frac{(-\cos x \sin x)(-1)}{(\cos x - \sin x)(-1)} && \text{Simplify and multiply by } \frac{-1}{-1} \\
&= \frac{\sin x \cos x}{\sin x - \cos x}
\end{aligned}$$

Example 3.2.5

Prove that $\frac{\tan^2 \theta + 2}{1 + \tan^2 \theta} = 1 + \cos^2 \theta$.

Solution: Expand the left side:

$$\begin{aligned}
\frac{\tan^2 \theta + 2}{1 + \tan^2 \theta} &= \frac{(\tan^2 \theta + 1) + 1}{1 + \tan^2 \theta} \\
&= \frac{\sec^2 \theta + 1}{\sec^2 \theta} && \text{by identity (3.6)} \\
&= \frac{\sec^2 \theta}{\sec^2 \theta} + \frac{1}{\sec^2 \theta} && \text{separate fractions.} \\
&= 1 + \cos^2 \theta && \text{reciprocal identity}
\end{aligned}$$

Example 3.2.6

Verify the identity $\frac{1}{\sec x \tan x} = \csc x - \sin x$

Solution: We will begin on the left side by converting to sines and cosines

$$\frac{1}{\sec x \tan x} = \frac{1}{\sec x} \cdot \frac{1}{\tan x} \quad \text{write as two fractions}$$

$$\begin{aligned}
&= \cos x \left(\frac{\cos x}{\sin x} \right) && \text{Convert to sine and cosine} \\
&= \frac{\cos^2 x}{\sin x} && \text{Simplify} \\
&= \frac{1 - \sin^2 x}{\sin x} && \text{identity (3.5)} \\
&= \frac{1}{\sin x} - \frac{\sin^2 x}{\sin x} && \text{Write as separate fractions} \\
&= \csc x - \sin x && \text{Reciprocal identities}
\end{aligned}$$

Example 3.2.7

Find all solutions to

$$\cos x + \sin x \tan x = 2$$

Solution: We need to be able to either factor this expression or write it in terms of a single trigonometric function. We saw in **Example 3.2.1** that this equation can be simplified to $\cos x + \sin x \tan x = \sec x$. Now we can solve it.

$$\sec x = 2 \quad \implies \quad \cos x = \frac{1}{2} \quad \implies \quad x = \frac{\pi}{3}, \frac{5\pi}{3}$$

The general solution is

$$x = \frac{\pi}{3} + 2n\pi, \quad x = \frac{5\pi}{3} + 2n\pi, \quad \text{where } n \in \mathbb{Z}$$

3.2 Exercises

For Exercises 1-6 simplify each expression to an expression involving a single trigonometric function with no fractions.

1. $\frac{1 + \tan x}{1 + \cot x}$

2. $\frac{1 + \csc t}{1 + \sin t}$

3. $\frac{1 - \sin^2 x}{1 + \sin x}$

4. $\frac{\sec \theta - \cos \theta}{\sin \theta}$

5. $\frac{\tan \theta}{\sec \theta - \cos \theta}$

6. $\frac{\sin x}{1 + \cos x} + \frac{\cos x}{\sin x}$

For Exercises 7 - 28, use trigonometric identities to show the identity is true. Remember, you may only work with one side of the equation at a time so do not cross multiply.

7. $\csc \theta (\sin \theta + \cos \theta) = 1 + \cot \theta$
8. $\cos \theta \sec \theta - \sin^2 \theta = \cos^2 \theta$
9. $\sec \alpha - \tan \alpha = \frac{\cos \alpha}{1 + \sin \alpha}$
10. $\frac{\cos^2 \beta - \sin^2 \beta}{1 - \tan^2 \beta} = \cos^2 \beta$
11. $\frac{1 - \tan^2 x}{1 + \tan^2 x} = 2 \cos^2 x - 1$
12. $\sec \theta - \frac{1}{\sec \theta} = \sin \theta \tan \theta$
13. $\frac{\sin \theta}{\cos \theta} + \frac{\cos \theta}{\sin \theta} = \csc \theta \sec \theta$
14. $\frac{\sin \theta}{1 + \sin \theta} - \frac{\sin \theta}{1 - \sin \theta} = -2 \tan^2 \theta$
15. $2 \tan x - (1 + \tan x)^2 = -\sec^2 x$
16. $\tan^2 \theta - 3 \sin \theta \tan \theta \sec \theta = -2 \tan^2 \theta$
17. $\frac{1}{\cos^2 x} - \frac{1}{\cot^2 x} = 1$
18. $\frac{\sin \theta}{\cos \theta} + \frac{\cos \theta}{\sin \theta} = \sec \theta \csc \theta$
19. $\tan x (\cot x - \cos x) = 1 - \sin x$
20. $\frac{\sec \theta \sin \theta}{\tan \theta} - 1 = 0$
21. $\frac{\cos^2 \theta}{1 + \sin \theta} = 1 - \sin \theta$
22. $\cos x = 1 - \frac{\sin^2 x}{1 + \cos x}$
23. $\frac{1 + \sin x}{\cos x} = \frac{\cos x}{1 - \sin x}$
24. $\tan^2 x = \frac{-\sin^2 x}{\sin^2 x - 1}$
25. $\sin^4 x - \cos^4 x = \sin^2 x - \cos^2 x$
26. $\csc x - \sin x = \cot x \cos x$
27. $\tan x - \cot x = \frac{1 - 2 \cos^2 x}{\sin x \cos x}$
28. $\cos \theta + \frac{\sin^2 \theta}{\cos \theta} = \sec \theta$

For Exercises 29 - 34, use trigonometric identities to simplify each equation, then find all solutions on $[0, 2\pi)$. Leave your answers in radians.

29. $\cos^2 x \tan^2 x = 1$
30. $\sin \theta = \cos \theta$
31. $2 \cos^2 x - \sin x - 1 = 0$
32. $\cos^2 x = -6 \sin x$
33. $\tan x - 3 \sin x = 0$
34. $2 \tan^2 \theta = 3 \sec \theta$