Understand these concepts from Chapters 5 and 6:

1. Identify whether a set of vectors forms a basis for a set.
2. Know that the dimension of a vector space is equal to the number of vectors in its basis.
3. Know what the rank and nullity of a matrix are.
4. What eigenvalues and eigenvectors represent. Solutions to $Ax = \lambda x$
5. How to calculate the eigenvalues of a small matrix (2×2 or 3×3) by hand.
6. How to calculate the eigenvectors of a matrix.
7. What the algebraic multiplicity of an eigenvalue is.
8. Chapter 5, Theorem 2: If $\{v_1, v_2, \ldots, v_r\}$ are eigenvectors which correspond to distinct eigenvalues, then the set $\{v_1, v_2, \ldots, v_r\}$ is linearly independent.
9. How to write the characteristic equation of a matrix and what it is used for.
10. How to diagonalize a matrix. When would diagonalization be advantageous?
11. How to add, subtract, and multiply complex numbers.
12. How to find the complex eigenvalues and eigenvectors of a matrix.
13. Compute the dot product of a pair of vectors.
14. Compute the norm of a vector.
15. Determine the angle between a pair of vectors.
16. Identify orthogonal vectors and sets.
17. Find the projection of a vector onto a subspace.
18. Find the shortest distance from a vector to subspace.
19. Given an orthogonal basis $\{u_1, u_2, \ldots, u_r\}$ for a subspace W write vector $y \in W$ as $y = c_1u_1 + c_2u_2 + \cdots + c_ru_r$
20. Use the Gram-Schmidt process to construct an orthonormal basis for subspace W.

Practice Problems

1. Identify whether the following statements are true or false. If a statement is false, give an explanation why.

   (a) An $n \times n$ matrix has n eigenvalues.

   (b) $\dfrac{6-12i}{2+3i} = 3-4i$

   (c) If A is similar to the identity matrix, then $\det A = 0$.

   (d) If x is an eigenvector of A, then the line through the origin and x passes through Ax.

   (e) If B and C are bases for the same vector space V, then B and C contain the same number of vectors.

   (f) If u and v are in $\mathbb{R}^n$, then $u \cdot v = v \cdot u$.

   (g) A is a diagonalizable matrix if $A = PDP^{-1}$ for some matrix D and some invertible matrix P.

2. You are given that

$$x_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix},\ x_2 = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix},\ x_3 = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix},\ P = \begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix}, D = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & -4 \end{bmatrix}$$

   and $A = PDP^{-1}$. Then $A^2 x_2 =$

   (a) $\begin{bmatrix} 25 \\ 0 \\ 50 \end{bmatrix}$ (b) $\begin{bmatrix} 25 \\ 0 \\ 100 \end{bmatrix}$ (c) $\begin{bmatrix} 2 \\ 0 \\ -8 \end{bmatrix}$ (d) $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ (e) $\begin{bmatrix} 5 \\ 0 \\ 20 \end{bmatrix}$

3. Find the eigenvalues and eigenvectors of $A = \begin{bmatrix} -4 & 6 \\ -1 & 1 \end{bmatrix}$

4. Find the eigenvalues and eigenvectors of $A = \begin{bmatrix} -3 & 4 \\ 3 & 8 \end{bmatrix}$

5. Find the eigenvalues and eigenvectors of $A = \begin{bmatrix} 3 & -5 \\ 9 & 3 \end{bmatrix}$

6. Let $A = \begin{bmatrix} 4 & 3 & 2 \\ 2 & 9 & 4 \\ -1 & -3 & 1 \end{bmatrix}$. THe eigenvalues of A are $\lambda_1 = 3$ and $\lambda_2 = 8$. Find an eigenbasis for each eigenvalue.

7. The 3×3 matrix A has eigenvalues $\lambda_1 = 3$, $\lambda_2 = -4$, $\lambda_3 = 1$, with corresponding eigenvectors $v_1 = \begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix}$, $v_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$, $v_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$. Write A in the form PDP^{-1}, where D is a diagonal matrix.

8. Let $A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 2 \\ 0 & 1 & 1 \end{bmatrix}$. Which of the following vectors is orthogonal to the column space of A?

   (a) $\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$ (b) $\begin{bmatrix} 1 \\ 4 \\ 2 \end{bmatrix}$ (c) $\begin{bmatrix} 0 \\ 1 \\ -2 \end{bmatrix}$ (d) $\begin{bmatrix} 2 \\ 0 \\ 2 \end{bmatrix}$

   (e) Since the second row is a multiple of the third row, Col A is undefined.

9. Let $A = \begin{bmatrix} 4 & 2 & -3 \\ 3 & 4 & 1 \\ 4 & 1 & 5 \end{bmatrix}$. Then $\lambda = 3$ is an eigenvalue corresponding to the eigenvector $v = \begin{bmatrix} 1 \\ -2 \\ a \end{bmatrix}$. Find the value of a.

10. Let u, v be vectors in $\mathbb{R}^n$ with θ the angle between u and v. Then $||u+v||^2$ is equal to:

    (a) $||u||^2||v||^2 \cos\theta$

    (b) $||u||^2 + ||v||^2$

    (c) $||u||^2 - ||v||^2$

    (d) $||u||^2 + ||v||^2 + 2||u||^2||v||^2 \cos\theta$

    (e) $||u||^2 + ||v||^2 - 2||u||^2||v||^2 \cos\theta$

11. Let

$$A = \begin{bmatrix} 5 & -1 & 3 & -1 \\ 0 & 4 & h & 0 \\ 0 & 0 & 5 & 4 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

    Find h so that the eigenspace corresponding to the eigenvalue $\lambda = 5$ is 2-dimensional.

12. Suppose the 2×2 matrix A has eigenvalues $\lambda_1 = 4$ and $\lambda_2 = 3$ with eigenvectors v_1 and v_2, respectively. If $u = 5v_1 + v_2$, then A^2u is equal to

    (a) $25v_1 + v_2$

    (b) $25v_1 + 3v_2$

    (c) $80v_1 + 9v_2$

    (d) $100v_1 + 3v_2$

    (e) $400v_1 + 9v_2$

13. An $n \times n$ matrix B has characteristic polynomial $p(\lambda) = -\lambda(\lambda - 3)^3(\lambda - 2)^2(\lambda + 1)$. Which of the following statements is **FALSE**?

    (a) rank $B = 6$.

    (b) det $(B) = 0$.

    (c) det $(B^T B) = 0$.

    (d) B is invertible.

    (e) $n = 7$.

14. Given vectors $u = \begin{bmatrix} 10 \\ 0 \\ 5 \end{bmatrix}$ and $v = \begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix}$ find $u \cdot v$.

15. Find a unit vector in the direction of $v = \begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix}$

16. Find the distance between the two vectors. $u = (6, -12), v = (-12, 12)$

17. Determine whether the set of vectors is orthogonal.

$$\begin{bmatrix} -2 \\ -4 \\ -2 \end{bmatrix}, \begin{bmatrix} 20 \\ 0 \\ -20 \end{bmatrix}, \begin{bmatrix} -20 \\ -20 \\ -20 \end{bmatrix}$$

18. Find the orthogonal projection of $y = \begin{bmatrix} -24 \\ 10 \end{bmatrix}$ onto $u = \begin{bmatrix} 4 \\ 20 \end{bmatrix}$.

19. Let W be the subspace spanned by $u_1 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$ and $u_2 = \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}$. Write $y = \begin{bmatrix} 19 \\ 3 \\ 11 \end{bmatrix}$ as the sum of a vector in W and a vector orthogonal to W. Find the closest point to y in the subspace W and the shortest distance.

Practice Problems Answers

1. Identify whether the following statements are true or false. If a statement is false, give an explanation why.
   (a) True, including multiplicity.
   (b) False.
   (c) False, $\det A = 1$.
   (d) True.
   (e) True
   (f) True.
   (g) True.

2. $\begin{bmatrix} 25 \\ 0 \\ 50 \end{bmatrix}$

3. $E_{\lambda=-2} = \left\{ \begin{bmatrix} 3 \\ 1 \end{bmatrix} \right\}$ and $E_{\lambda=-1} = \left\{ \begin{bmatrix} 2 \\ 1 \end{bmatrix} \right\}$

4. $E_{\lambda=9} = \left\{ \begin{bmatrix} 1 \\ 3 \end{bmatrix} \right\}$ and $E_{\lambda=-4} = \left\{ \begin{bmatrix} -4 \\ 1 \end{bmatrix} \right\}$

5. $E_{\lambda=3+3i\sqrt{5}} = \left\{ \begin{bmatrix} i\sqrt{5} \\ 3 \end{bmatrix} \right\}$

6. $E_{\lambda=3} = \left\{ \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -3 \\ 1 \\ 0 \end{bmatrix} \right\}$ and $E_{\lambda=8} = \left\{ \begin{bmatrix} -1 \\ -2 \\ 1 \end{bmatrix} \right\}$

7. $A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 3 & 1 & 1 \end{bmatrix} \begin{bmatrix} 3 & 0 & 0 \\ 0 & -4 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -2 & -1 & 1 \end{bmatrix}$

8. $\begin{bmatrix} 0 \\ 1 \\ -2 \end{bmatrix}$

9. $a = -1$.

10. $||u||^2 + ||v||^2 + 2||u||^2||v||^2 \cos\theta$

11. $h = 3$

12. $80v_1 + 9v_2$

13. B is invertible is the False statement because $\lambda = 0$ can only be an eigenvalue of a singular matrix.

14. $u \cdot v = 15$.

15. unit vector $= \begin{bmatrix} 2/\sqrt{14} \\ 3/\sqrt{14} \\ -1/\sqrt{14} \end{bmatrix}$

16. 30

17. No

18. $\text{proj}_u y = \begin{bmatrix} 1 \\ 5 \end{bmatrix}$.

19. $y = \begin{bmatrix} 18 \\ 7 \\ 10 \end{bmatrix} + \begin{bmatrix} 1 \\ -4 \\ 1 \end{bmatrix}$. Closest point $(18, 7, 10)$. Shortest distance $\sqrt{18}$.