3.11 Higer Order Linear Homogeneous Differential Equations

Everything is mostly the same only now we have

$$\frac{d^n y}{dt^n} + p_{n-1}(t)\frac{d^{n-1}y}{dt^{n-1}} + \cdots + p_1(t)\frac{dy}{dt} + p_0(t)y = g(t)$$

Existence and Uniqueness:

Let $p_{n-1}(t), \cdots, p_1(t), p_0(t)y$ and $g(t)$ be continuous functions defined on the interval $a < t < b$, and let t_0 be in (a, b). Then the initial value problem

$$\frac{d^n y}{dt^n} + p_{n-1}(t)\frac{d^{n-1}y}{dt^{n-1}} + \cdots + p_1(t)\frac{dy}{dt} + p_0(t)y = g(t)$$
$$y(t_o) = y_o, y'(t_o) = y_o', y''(t_o) = y_o'', \ldots, y^{(n-1)}(t_o) = y_o^{(n-1)}$$

has a unique solution defined on the entire interval (a, b).

We need n initial conditions to solve an IVP here. As in the 2nd order case we will find a fundamental set of solutions to the homogeneous equation. We can still determine if we have a fundamental set by calculating the Wronskian

$$W = \begin{vmatrix} y_1 & y_2 & \cdots & y_n \\ y_1' & y_2' & \cdots & y_n' \\ \vdots & \vdots & \ddots & \vdots \\ y_1^{(n-1)} & y_2^{(n-1)} & \cdots & y_n^{(n-1)} \end{vmatrix}$$

If $W \neq 0$ then $y_1, y_2, \cdots y_n$ form a fundamental set of solutions for
$\frac{d^n y}{dt^n} + p_{n-1}(t)\frac{d^{n-1}y}{dt^{n-1}} + \cdots + p_1(t)\frac{dy}{dt} + p_0(t)y = 0$.

For the nonhomogeneous case there is still only one particular solution y_p and the total solution is $Y = y_h + y_p$

Theorem 3.6 (Abel's Theorem) : Let $y_1(t),\ y_2(t), \cdots y_n(t)$ be *n* solutions of the homogenous linear differential equation

$$\frac{d^n y}{dt^n} + p_{n-1}(t)\frac{d^{n-1}y}{dt^{n-1}} + \cdots + p_1(t)\frac{dy}{dt} + p_0(t)y = 0, \quad a < t < b,$$

Where $p_0(t),\ p_1(t), \cdots p_{n-1}(t)$ are continuous on (*a*, *b*). Let $W(t)$ be the Wronskian of $y_1(t),\ y_2(t), \cdots y_n(t)$. If t_0 is any point in (*a*, *b*), then

$$W(t) = W(t_0)e^{-\int_{t_0}^{t} p_{n-1}(s)ds}, \quad a < t < b.$$

Why do we care about Theorem 3.6? Because it says that if the Wronskian is not zero at **any** point of (*a*, *b*) then it is not zero at **every** point of (*a*, *b*).

Linear Independence

Definition: If $ax_1 + bx_2 = 0$ where *a* and *b* are not both zero then x_1 and x_2 are said to be Linearly Dependent. x_1 and x_2 are Linearly Independent otherwise.

The idea of linear independence applies to both functions and vectors. For functions we talk about linear independence on an interval.

If $ax_1(t) + bx_2(t) = 0$ with *a* and *b* not both zero for all *t* in *I* then $x_1(t)$ and $x_2(t)$ are said to be Linearly Dependent.

Easier Way: (Sometimes)

Theorem: If *f* and *g* are differentiable functions on an open interval *I* and if $W(f,g)(t_0) \neq 0$ **for some point** t_0 in *I*, the *f* and *g* are linearly independent on *I*.
Furthermore if *f* and *g* are linearly dependent on *I*, then $W(f,g)(t_0) = 0$ for all *t* in *I*.

Ex 1: Consider the differential equation $y''+2ty'+t^2y=0$ on the interval $-\infty < t < \infty$. Assume that y_1 and y_2 are two solutions satisfying the given initial conditions.

(a) Do the solutions form a fundamental set?
(b) Do the two solutions form a linearly independent set of functions on $-\infty < t < \infty$

1) $y_1(1)=2$, $y_1'(1)=2$; $y_2(1)=-1$, $y_2'(1)=-1$

2) $y_1(0)=0$, $y_1'(0)=1$; $y_2(0)=-1$, $y_2'(0)=0$

Ex 2: Find a (3x3) constant matrix A such that $[\bar{y}_1(t),\ \bar{y}_2(t), \bar{y}_3(t)] = [y_1(t),\ y_2(t), y_3(t)]A$. Then determine whether $\{\bar{y}_1(t),\ \bar{y}_2(t), \bar{y}_3(t)\}$ is also a fundamental set by calculating det(A).

$y''' - y'' = 0$, $\{y_1(t),\ y_2(t), y_3(t)\} = \{1, t, e^{-t}\}$
$\{\bar{y}_1(t),\ \bar{y}_2(t), \bar{y}_3(t)\} = \{1-2t, t+2, e^{-(t+2)}\}$