3.9 Variation of Parameters: (The plow method)

Recall: There are always 3 steps to solving $ay''+by'+cy = g(t)$

Step 1: Find the homogeneous solution $y_h = C_1 y_1 + C_2 y_2$.

Step 2: Find the particular solution Y_P

Step 3: Add them together $y = y_h + Y_P = C_1 y_1 + C_2 y_2 + Y_P$

In section 4.9 we saw that we could guess at a solution that looked like the answer. That is fine if your answer is nice but it doesn't always work well. Variation of parameters is a completely general form that applies to all situations. However, it isn't always possible to solve the problem explicitly because in the end there are always integrals to be evaluated.

Variation of Parameters:

Start with $y''+p(t)y'+q(t)y = g(t)$

Step 1: Solve the homogeneous equation for the family of solutions $\{y_1, y_2\}$.

Step 2: Let the particular solution have the form:

$$y_P = v(t)y_1 + u(t)y_2$$

To solve for *u* and *v* we need to find y_P' and y_P'' and substitute back into the original equation.

$$y_P' = v\,y_1' + v'\,y_1 + u\,y_2' + u'\,y_2$$

Now at this point we really have generated 4 variables so we need to put some constraints on the system. There are many choices we could make here but the best choice is to assume that

$$v'\,y_1 + u'\,y_2 = 0 \qquad (1)$$

Then we can reduce to $y_P' = v\,y_1' + u\,y_2'$ and take a second derivative

$$y_P'' = v\,y_1'' + v'\,y_1' + u\,y_2'' + u'\,y_2'$$

Then we plug back into the original equation:

$$v\,y_1'' + v'\,y_1' + u\,y_2'' + u'\,y_2' + p(t)\left(v\,y_1' + u\,y_2'\right) + q(t)\left(v\,y_1 + u\,y_2\right) = g(t)$$

This equation can be simplified

$$v\,y_1''+p(t)v\,y_1'+q(t)v\,y_1+u\,y_2''+p(t)u\,y_2'+q(t)u\,y_2+v'\,y_1'+u'\,y_2'=g(t)$$

$$v'\,y_1'+u'\,y_2'=g(t) \qquad (2)$$

Now solve (1) and (2) for *u* and *v*. Then your solution is once again of the form:

$$\boxed{y=y_h+y_p}$$

Ex 1: $y''+9y=9\sec^2 3t$

$$r^2+9=0 \text{ so } r=\pm 3i$$

and

$$y_h=C_1\cos 3t+C_2\sin 3t$$

For the particular solution choose

$$y_p=v\cos 3t+u\sin 3t$$

Ex 2: $ty'' - (1+t)y' + y = t^2 e^{2t}$ and the homogeneous solutions are $y_1 = 1+t$ and $y_2 = e^t$.

For the particular solution choose:

$$y_p = v(1+t) + u(e^t)$$