3.2 The General Solutions of Homogeneous Equations

We will start by looking as solutions of the homogeneous equation

$$y''+p(t)y'+q(t)y=0, \qquad a<t<b,$$

Where $p(t)$ and $q(t)$ are continuous on (a, b).

Definition: Let $f_1(t)$ and $f_2(t)$ be any two functions having a common domain, and let c_1 and c_2 be any two constants. Then the function $F(t)=c_1f_1(t)+c_2f_2(t)$ is a **linear combination** of $f_1(t)$ and $f_2(t)$. We can extend the definition in the obvious way to describe any number of functions.

Theorem 3.2: The principle of superposition:
If $y_1(t)$ and $y_2(t)$ are solutions to $y''+p(t)y'+q(t)y=0$ defined on the interval $a<t<b$, where $p(t)$ and $q(t)$ are continuous on (a, b). Then the linear combination $y(t)=C_1y_1(t)+C_2y_2(t)$ is also a solution of the differential equation.

Pf: $y'=y_1'(t)+y_2'(t)$
$y''=y_1''(t)+y_2''(t)$

Substitute back into $y''+p(t)y'+q(t)y$ to get:

$$(y_1''(t)+y_2''(t))+p(t)\left(y_1'(t)+y_2'(t)\right)+g(t)(y_1''(t)+y_2''(t))$$
$$=\underbrace{y_1''+p(t)y_1'+g(t)y_1}_{=0}+\underbrace{y_2''+p(t)y_2'+g(t)y_2}_{=0}=0 \qquad \text{q.e.d.}$$

If $y_1(t)$ and $y_2(t)$ are two solutions of $y''+p(t)y'+q(t)y=0$ and every other solution $y(t)$ can be written as a linear combination of these two (ie. $y(t)=C_1y_1(t)+C_2y_2(t)$) then $y_1(t)$ and $y_2(t)$ are a **fundamental set of solutions**.

Consider $y''+p(t)y'+q(t)y=0$ with the initial conditions $y(t_0)=y_0,\ y'(t_0)=y_0'$. If we start with 2 solutions y_1 and y_2 then $y=C_1y_1+C_2y_2$ is also a solution and by using the initial condition we get a system of equations:

$$\begin{aligned} y_0 &= C_1y_1(t_0)+C_2y_2(t_0) \\ y_0' &= C_1y_1'(t_0)+C_2y_2'(t_0) \end{aligned}$$

We can use Cramer's Rule to solve the system:

$$C_1=\frac{\begin{vmatrix} y_0 & y_2(t_0) \\ y_0' & y_2'(t_0) \end{vmatrix}}{\begin{vmatrix} y_1(t_0) & y_2(t_0) \\ y_1'(t_0) & y_2'(t_0) \end{vmatrix}} \quad \text{and} \quad C_2=\frac{\begin{vmatrix} y_1(t_0) & y_0 \\ y_1'(t_0) & y_0' \end{vmatrix}}{\begin{vmatrix} y_1(t_0) & y_2(t_0) \\ y_1'(t_0) & y_2'(t_0) \end{vmatrix}}$$

Notice that C_1 and C_2 have solutions as long as

$$\begin{vmatrix} y_1(t_0) & y_2(t_0) \\ y_1'(t_0) & y_2'(t_0) \end{vmatrix} \neq 0$$

or $\quad y_1 y_2' - y_2 y_1' \neq 0$ at t_0

This determinant is known as the Wronskian: $W = \begin{vmatrix} y_1(t_0) & y_2(t_0) \\ y_1'(t_0) & y_2'(t_0) \end{vmatrix}$

Theorem 3.3: Suppose y_1 and y_2 are two solutions to

$$y'' + p(t)y' + q(t)y = 0, \quad a < t < b,$$

Where $p(t)$ and $q(t)$ are continuous on (a, b). Let $W(t)$ be the Wronskian of y_1 and y_2. If there is a point t_0 in (a, b) such that $W(t_0) \neq 0$, then { y_1, y_2 } is a fundamental set of solutions.

Theorem 3.4: Let y_1 and y_2 be solutions of the homogenous linear differential equation

$$y'' + p(t)y' + q(t)y = 0, \quad a < t < b,$$

Where $p(t)$ and $q(t)$ are continuous on (a, b). Let $W(t)$ be the Wronskian of y_1 and y_2. If t_0 is any point in (a, b), then

$$W(t) = W(t_0) e^{-\int_{t_0}^{t} p(s)ds}$$

Why do we care about Theorem 4.4? Because it says that if the Wronskian is not zero at **any** point of (a, b) then it is not zero at **every** point of (a, b).

Ex:

(a) Determine whether the given functions are solutions of the differential equation.
(b) If both functions are solutions, calculate the Wronskian. Does this calculation show that the two functions form a fundamental set of solutions?
(c) If the two functions have been shown in part (b) to form a fundamental set, construct the general solution and determine the unique solution satisfying the given initial conditions.

1. $y'' + y = 0$; $y_1(t) = \sin t \cos t$, $y_2(t) = \sin t$; $y(\pi/2) = 1$ and $y'(\pi/2) = 1$

2. $y''-4y'+4y=0$; $y_1(t)=e^{2t}$, $y_2(t)=te^{2t}$; $y(0)=2$ and $y'(0)=0$

3. $ty''+y'=0$, $0<t<\infty$; $y_1(t)=\ln t$, $y_2(t)=\ln(3t)$; $y(3)=0$ and $y'(3)=3$

4. $4y''+y=0$; $y_1(t)=\sin((t/2)+(\pi/3))$, $y_2(t)=\sin((t/2)-(\pi/3))$; $y(0)=0$ and $y'(0)=1$