System transfer functions

$$\hat{Y}(s) = \phi(s)\hat{F}(s)$$

$$y(t) = \mathcal{L}^{-1}\{\phi(s)\hat{F}(s)\}$$

Convolution integral:

Two functions $f(t)$ & $g(t)$ with Laplace transforms $F(s)$ & $G(s)$

$$\underline{\mathcal{L}^{-1}\{F(s)G(s)\}} = (f \overbrace{*}^{\text{convolution}} g)(t)$$

$$= \int_0^t f(t-\lambda)\, g(\lambda)\, d\lambda$$

Ex 1: $e^{3t} * e^{-t} = \int_0^t e^{3(t-\lambda)} e^{-\lambda} d\lambda$

$$= \int_0^t e^{3t-3\lambda-\lambda} d\lambda$$

$$= -\frac{1}{4}\int_0^t e^{3t-4\lambda}(-4\,d\lambda)$$

$u = 3t - 4\lambda$
$\underbrace{du = -4\,d\lambda}_{\lambda}$

$$= -\frac{1}{4}\left[e^{3t-4\lambda}\right]_{\lambda=0}^{\lambda=t}$$

$$\boxed{e^{3t} * e^{-t} = -\frac{1}{4}\left(e^{-t} - e^{3t}\right)}$$

$\mathcal{L}\{e^{\alpha t}\} = \frac{1}{s-\alpha}$

OR $f(t) = e^{3t}$ $g(t) = e^{-t}$

OR $f(t) = e^{3t}$ $\quad g(t) = e^{-t}$

$F(s) = \frac{1}{s-3}$ $\quad G(s) = \frac{1}{s+1}$

$$e^{3t} * e^{-t} = \mathcal{L}^{-1}\left\{\frac{1}{s-3}\cdot\frac{1}{s+1}\right\} = \mathcal{L}^{-1}\left\{F(s)G(s)\right\}$$

Partial Fractions $\frac{1}{(s-3)(s+1)} = \frac{A}{s-3} + \frac{B}{s+1}$

$A = \frac{1}{4}$ $\quad B = -\frac{1}{4}$

$$e^{3t} * e^{-t} = \mathcal{L}^{-1}\left\{\frac{1}{4}\frac{1}{(s-3)} - \frac{1}{4}\frac{1}{s+1}\right\}$$

$$\boxed{= \frac{1}{4}e^{3t} - \frac{1}{4}e^{-t}}$$

Ex: $t * \cos(5t) = \int_0^t (t-\lambda)\cos(5\lambda)\,d\lambda$

$f(t) = t$ $\quad g(t) = \cos(5t)$

Parts

$u = t-\lambda$ $\quad dv = \cos(5\lambda)\,d\lambda$

$du = -d\lambda$ $\quad v = \frac{1}{5}\sin(5\lambda)$

$$= \frac{(t-\lambda)}{5}\sin(5\lambda) + \int_0^t \frac{1}{5}\sin(5\lambda)\,d\lambda$$

$$= \left(\frac{t-\lambda}{5}\right)\sin 5\lambda - \frac{1}{25}\cos(5\lambda)\Big|_0^t$$

$$= \cancel{\left(\frac{t-t}{5}\right)}^{0}\sin(5t) - \frac{1}{25}\cos(5t) - \left[\cancel{\frac{t}{5}\sin(0)}^{0} - \frac{1}{25}\cos 0\right]$$

$$\boxed{= -\frac{1}{25}\cos(5t) + \frac{1}{25}} = \mathcal{L}^{-1}\left\{\frac{1}{s^2}\,\frac{s}{s^2+5^2}\right\}$$