Laplace Transform Table

Friday, April 26, 2019 8:52 AM

Laplace_transforms

Table 1: **A Brief Table Of Laplace Transform Pairs**

$f(t)$, $t \geq 0$	Laplace Transform $F(s)$	
1. $h(t) = \begin{cases} 1 & t \geq 0 \\ 0, & t < 0 \end{cases}$	$\dfrac{1}{s}$	$(s > 0)$
2. 1	$\dfrac{1}{s}$	$(s > 0)$
3. $t^n, \qquad n = 1, 2, 3, \ldots$	$\dfrac{n!}{s^{n+1}}$	$(s > 0)$
4. $e^{\alpha t}$	$\dfrac{1}{s-\alpha}$	$(s > \alpha)$
5. $\sin \omega t$	$\dfrac{\omega}{s^2 + \omega^2}$	$(s > 0)$
6. $\cos \omega t$	$\dfrac{s}{s^2 + \omega^2}$	$(s > 0)$
7. $\sinh bt = \dfrac{e^{bt} - e^{-bt}}{2}$	$\dfrac{b}{s^2 - b^2}$	$(s > \lvert b \rvert)$
8. $\cosh bt = \dfrac{e^{bt} + e^{-bt}}{2}$	$\dfrac{s}{s^2 - b^2}$	$(s > \lvert b \rvert)$
9. $e^{\alpha t} f(t)$, with $\lvert f(t) \rvert \leq M e^{at}$	$F(s - \alpha)$	$s > \alpha + a$
(10) - (13) are four special cases of (9)		
10. $e^{\alpha t} h(t)$	$\dfrac{1}{s-\alpha}$	$(s > \alpha)$
11. $e^{\alpha t} t^n, \qquad n = 1, 2, 3, \ldots$	$\dfrac{n!}{(s-\alpha)^{n+1}}$	$(s > \alpha)$
12. $e^{\alpha t} \sin \omega t$	$\dfrac{\omega}{(s-\alpha)^2 + \omega^2}$	$(s > \alpha)$
13. $e^{\alpha t} \cos \omega t$	$\dfrac{s-\alpha}{(s-\alpha)^2 + \omega^2}$	$(s > \alpha)$
14. $f(t-\alpha) h(t-\alpha), \qquad (\alpha \geq 0),$ with $\lvert f(t) \rvert \leq M e^{at}$	$e^{-\alpha s} F(s)$	$(s > a)$
15. $h(t - \alpha), \qquad \alpha \geq 0$	$\dfrac{e^{-\alpha s}}{s}$	$(s > 0)$
16. $f'(t)$, with $f(t)$ continuous with $\lvert f'(t) \rvert \leq M e^{at}$	$sF(s) - f(0)$	$s > \max\{a, 0\}$
17. $f''(t)$, with $f'(t)$ continuous with $\lvert f''(t) \rvert \leq M e^{at}$	$s^2 F(s) - s f(0) - f'(0)$	$s > \max\{a, 0\}$

18. $f^{(n)}(t)$, with $f^{(n-1)}(t)$ continuous with $\lvert f^{(n)}(t)\rvert \le Me^{at}$	$s^nF(s) - s^{n-1}f(0) - \cdots$ $-sf^{(n-2)} - f^{(n-1)}(0)$	$s > \max\{a, 0\}$
19. $\displaystyle\int_0^t f(u)\,du$, with $\lvert f^{(n)}(t)\rvert \le Me^{at}$	$\dfrac{F(s)}{s}$	$s > \max\{a, 0\}$
20. $\displaystyle\int_0^t f(t-\lambda)g(\lambda)\,d\lambda$	$F(s)\cdot G(s)$	
21. $t\cos\omega t$	$\dfrac{s^2-\omega^2}{(s^2+\omega^2)^2}$	$(s > 0)$
22. $t\sin\omega t$	$\dfrac{2\omega s}{(s^2+\omega^2)^2}$	$(s > 0)$
23. $\sin\omega t - \omega t\cos\omega t$	$\dfrac{2\omega^3}{(s^2+\omega^2)^2}$	$(s > 0)$
24. $\omega t - \sin\omega t$	$\dfrac{\omega^3}{s^2(s^2+\omega^2)^2}$	$(s > 0)$
25. $tf(t)$	$-F'(s)$	
26. $t^kf(t)$	$(-1)^kF^{(k)}(s)$	
27. $\delta(t-a)$	e^{-as}	$(s > 0)$

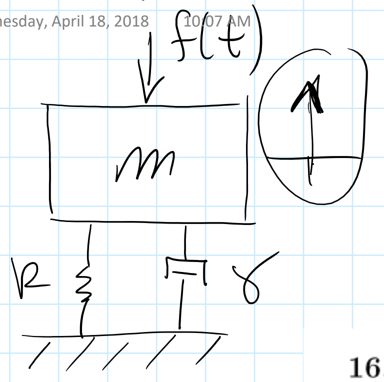
For Periodic Functions:

Let $f(t)$ be a piecewise continuous periodic function defined on $0 \le t < \infty$, where $f(t)$ has period T. Then

$$\mathcal{L}\{f(t)\} = \frac{\displaystyle\int_0^T e^{-st}f(t)\,dt}{1-e^{-sT}}, \qquad s > 0$$

Convolution

$$(f * g)(t) = \int_0^t f(t-\lambda)g(\lambda)\,d\lambda$$



$$my'' + \gamma y' + ky = f(t)$$

Solve with Laplace

$y'(0) = 0$
$y(0) = 0$

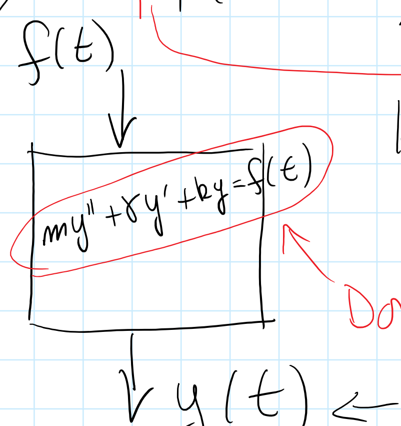
16. $f'(t)$, with $f(t)$ continuous with $\lvert f'(t)\rvert \le Me^{at}$	$sF(s) - f(0)$
17. $f''(t)$, with $f'(t)$ continuous with $\lvert f''(t)\rvert \le Me^{at}$	$s^2F(s) - sf(0) - f'(0)$

$$\mathcal{L}\{my'' + \gamma y' + ky\} = \mathcal{L}\{f(t)\}$$

$$ms^2Y(s) + \gamma sY(s) + kY(s) = F(s)$$

$$Y(s)(ms^2 + \gamma s + k) = F(s)$$

$$\boxed{Y(s) = \frac{1}{ms^2 + \gamma s + k}F(s)}$$



know input & output

$f(t)$ $\qquad$ $y(t)$

Don't know this information.

$$Y(s) = \phi(s)F(s)$$

$$\phi(s) = \frac{1}{ms^2 + \gamma s + k} = \frac{Y(s)}{F(s)}$$

System transfer function

$$Y(s) = \phi(s)\, F(s)$$

$$\phi(s) = \frac{1}{ms^2 + \gamma s + k} = \frac{Y(s)}{F(s)}$$

System transfer function

Know

$f(t)$

$\downarrow$

$\boxed{}$

$\downarrow y(t)$

Find $\phi = \dfrac{Y(s)}{F(s)}$

Do

$g(t)$

$\downarrow$

$\boxed{Y(s) = \phi(s)\, G(s)}$

$\downarrow$

$y(t)$ New answer.

Ex: Know we have a SMD system

$$my'' + \gamma y' + ky = f(t) \qquad y(0)=0,\ y'(0)=0$$

If we apply $f(t) = h(t)$ then the output is $y(t) = \frac{1}{2} - \frac{1}{2}e^{-t}\cos t - \frac{1}{2}e^{-t}\sin t$

Find $\hat{y}(t)$ for an input $\hat{f}(t) = e^{-2t}$

$$F(s) = \mathcal{L}\{f(t)\} = \mathcal{L}\{h(t)\} = \frac{1}{s} \qquad \text{row } \#1$$

$$Y(s) = \mathcal{L}\left\{\tfrac{1}{2}\right\} - \mathcal{L}\left\{\tfrac{1}{2}e^{-t}\cos t\right\} - \mathcal{L}\left\{\tfrac{1}{2}e^{-t}\sin t\right\}$$

row 13 row 12

$$= \frac{1}{2s} - \frac{1}{2}\frac{(s+1)}{(s+1)^2+1} - \frac{1}{2}\frac{1}{(s+1)^2+1}$$

$$= \frac{(s+1)^2+1}{2s((s+1)^2+1)} - \frac{s(s+1)}{2s((s+1)^2+1)} - \frac{s}{2s((s+1)^2+1)}$$

$$= \frac{s^2+2s+1+1-s^2-s-s}{2s((s+1)^2+1)} = \frac{1}{s((s+1)^2+1)}$$

$$\phi(s) = \frac{Y(s)}{F(s)} = \frac{\frac{1}{s((s+1)^2+1)}}{\frac{1}{s}} = \frac{1}{(s+1)^2+1} = \frac{1}{s^2+2s+2}$$

New $\hat{Y}(s)$ is

row 4 $\mathcal{L}\{e^{\alpha t}\} = \frac{1}{s-\alpha}$

$$\hat{Y}(s) = \phi(s)\,\hat{F}(s)$$

$$\hat{F}(s) = \mathcal{L}\{e^{-2t}\} = \frac{1}{s+2}$$

$$\hat{Y}(s) = \frac{1}{((s+1)^2+1)(s+2)}$$

Partial Fractions

$$\left[\frac{1}{((s+1)^2+1)(s+2)} = \frac{\overset{-\frac{1}{2}}{B}s+\overset{0}{\cancel{C}}}{(s+1)^2+1} + \frac{\overset{\frac{1}{2}}{\cancel{A}}}{(s+2)}\right](s+2)((s+1)^2+1)$$

$$1 = (Bs+C)(s+2) + A\left((s+1)^2+1\right)$$

$s=-2$ $\quad 1 = A(2) \Rightarrow A = \frac{1}{2}$

$s=0$ $\quad 1 = C(2) + \frac{1}{2}(2) \Rightarrow C = 0$

$s = 1 \qquad 1 = B(3) + \frac{1}{2}(5)$

$$\frac{2}{2} - \frac{5}{2} = 3B$$

$$-\frac{3}{2} = 3B \implies B = -\frac{1}{2}$$

$$\mathcal{L}^{-1}\{Y(s)\} = \mathcal{L}^{-1}\left\{ - \frac{s+1-1}{2\left((s+1)^2+1\right)} + \frac{1}{2(s+2)} \right\}$$

12. $e^{\alpha t}\sin\omega t$	$\dfrac{\omega}{(s-\alpha)^2+\omega^2}$
13. $e^{\alpha t}\cos\omega t$	$\dfrac{s-\alpha}{(s-\alpha)^2+\omega^2}$

$$y(t) = -\frac{1}{2}\mathcal{L}^{-1}\left\{\frac{s+1}{(s+1)^2+1}\right\} + \frac{1}{2}\mathcal{L}^{-1}\left\{\frac{1}{(s+1)^2+1}\right\} + \frac{1}{2}\mathcal{L}^{-1}\left\{\frac{1}{s+2}\right\}$$

$$\boxed{y(t) = -\frac{1}{2}e^{-t}\cos t + \frac{1}{2}e^{-t}\sin t + \frac{1}{2}e^{-2t}}$$

Section 5.6 Convolution

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System transfer functions

$$\hat{Y}(s) = \Phi(s)\,\hat{F}(s)$$

$$y(t) = \mathcal{L}^{-1}\{\Phi(s)\,\hat{F}(s)\}$$

<u>Convolution integral:</u>

Two functions $f(t)$ & $g(t)$ with Laplace transforms $F(s)$ & $G(s)$

$$\mathcal{L}^{-1}\{F(s)\,G(s)\} = (f \overbrace{*}^{\text{convolution}} g)(t)$$

$$= \int_0^t f(t-\lambda)\,g(\lambda)\,d\lambda$$

<u>Ex 1:</u> $e^{3t} * e^{-t} = \int_0^t e^{3(t-\lambda)} e^{-\lambda}\,d\lambda$

$$= \int_0^t e^{3t-3\lambda-\lambda}\,d\lambda$$

$$= -\frac{1}{4}\int_0^t e^{3t-4\lambda}(-4\,d\lambda) \qquad \begin{aligned} u &= 3t-4\lambda \\ du &= -4\,d\lambda \end{aligned}$$

$$= -\frac{1}{4}\Big[e^{3t-4\lambda}\Big]_{\lambda=0}^{\lambda=t}$$

$$\boxed{e^{3t} * e^{-t} = -\frac{1}{4}\left(e^{-t} - e^{3t}\right)}$$