Section 5.3 Partial Fractions

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Partial_fra...

$$\frac{2}{5} + \frac{3}{7} = \frac{2(7) + 3(5)}{5(7)} = \underbrace{\frac{29}{35}}_{\text{Start here}}$$

Denominator:	Partial fraction
linear $(s+a)$	$\frac{A}{s+a}$
repeated $(s+a)^n$	$\frac{A_1}{s+a} + \frac{A_2}{(s+a)^2} + \cdots + \frac{A_n}{(s+a)^n}$
quadratic s^2+as+b	$\frac{A_1 s + A_2}{s^2+as+b}$
repeated quadratic $(s^2+as+b)^2$	$\frac{A_1 s + A_2}{s^2+as+b} + \frac{A_3 s + A_4}{(s^2+as+b)^2}$

5.3 Partial Fraction Decomposition

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Partial Fractions consists of decomposing a rational function into simpler component fracti

Example 5.3.1. Denominator is a product of disctinct linear factors

$$\frac{3x+7}{x^2+6x+5} = \left(\frac{3x+7}{(x+5)(x+1)} = \frac{A}{x+5} + \frac{B}{x+1}\right)(x+5)(x+1)$$

$$3x+7 = A(x+1) + B(x+5)$$

$$3x+7 = \underbrace{Ax+Bx}_{} + A+5B$$

$$-(3 = A+B)$$

$$7 = A+5B$$

$$4 = 4B$$

$$1 = B$$

$$3 = A+1$$

$$2 = A$$

$$\boxed{\frac{3x+7}{(x+5)(x+1)} = \frac{2}{(x+5)} + \frac{1}{(x+1)}}$$

Example 5.3.2. Denominator is a product of linear factors, some of which are repeated.

$$\left(\frac{3x^2 - 8x + 13}{(x+3)(x-1)^2} = \frac{\cancel{A}\,4}{(x+3)} + \frac{\cancel{B}\,-1}{(x-1)} + \frac{\cancel{C}\,2}{(x-1)^2} \right)(x+3)(x-1)^2$$

$$3x^2 - 8x + 13 = \overset{4}{\cancel{A}}(x-1)^2 + B(x-1)(x+3) + \overset{2}{\cancel{C}}(x+3)$$

$x = 1$ $\quad 3 - 8 + 13 = 4C$

$8 = 4C$

$\underline{\underline{2 = C}}$

$x = -3$ $\quad 3(9) + 24 + 13 = A(16)$

$64 = 16A$

$\underline{\underline{4 = A}}$

$$\begin{array}{r} 27 \\ +24 \\ 13 \\ \hline 64 \end{array}$$

$x = 0$ $\quad 13 = 4 + B(-1)(3) + 6$

$3 = -3B$

$\underline{\underline{-1 = B}}$

Example 5.3.3. Denominator contains irreducible quadratic factors, none of which is repe

$$f(x) = \frac{2x^2+x-8}{x^3+4x} = \left(\frac{2x^2+x-8}{x(x^2+4)} = \frac{A}{x} + \frac{Bx+C}{x^2+4}\right) x(x^2+4)$$

$$2x^2+x-8 = \overset{-2}{\cancel{A}}(x^2+4) + (Bx+C)x$$

$$x=0 \qquad -8 = 4A$$

$$-2 = A$$

$$2x^2+x-8 = -2x^2-8+Bx^2+Cx$$

Derivative

$4x + 1 = -4x + 2Bx + C \qquad C = 1$

$4 = -4 + 2B$

$4 = B$

$$\boxed{f(x) = \frac{-2}{x} + \frac{4x+1}{x^2+4}}$$

Ex: $\mathcal{L}^{-1}\left\{ \dfrac{s^2+4}{s^4-s^2} \right\}$

Partial fractions

$$\left[\frac{s^2+4}{s^2(s-1)(s+1)} = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s-1} + \frac{D}{s+1} \right] s^2(s-1)(s+1)$$

$$s^2+4 = A\,s(s-1)(s+1) + B(s-1)(s+1) + C(s+1)s^2 + D\,s^2(s-1)$$

$s=0 \qquad 4 = -B \implies B = -4$

$s=1 \qquad 5 = 2C \implies C = 5/2$

$s=-1 \qquad 5 = -2D \implies D = -5/2$

$s=2 \qquad 8 = A(2)(1)(3) - 4(1)(3) + \frac{5}{\cancel{2}}(3)\overset{2}{\cancel{(4)}} - \frac{5}{\cancel{2}}\overset{2}{\cancel{(4)}}(1)$

$$8 = 6A - 12 + 30 - 10$$

$$8 = 6A + 8$$

$8 = 6A + 0$

$0 = A$

$$\mathcal{L}^{-1}\left\{ \frac{-4}{s^2} + \frac{5/2}{s-1} + \frac{-5/2}{s+1} \right\}$$

row 3   row4   row4

$$= -4\,\mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\} + \frac{5}{2}\,\mathcal{L}^{-1}\left\{\frac{1}{s-1}\right\} - \frac{5}{2}\,\mathcal{L}^{-1}\left\{\frac{1}{s+1}\right\}$$

$$\boxed{= -4t + \frac{5}{2}e^{t} - \frac{5}{2}e^{-t}}$$

<u>Ex</u>: Find inverse Laplace transform of

$$F(s) = \frac{50\,s}{(s+1)^2\,(s^2+4s+13)}$$

$$\frac{50s}{(s+1)^2(s^2+4s+13)} = \frac{A}{s+1} + \frac{B}{(s+1)^2} + \frac{Cs+D}{s^2+4s+13}$$

$$50s = A(s+1)(s^2+4s+13) + \overset{-5}{\cancel{B}}(s^2+4s+13) + (Cs+D)(s+1)^2$$

$s=-1$

$$-50 = B(1-4+13)$$
$$-50 = 10B$$
$$-5 = B$$

Derivative $\frac{d}{ds}[\quad]$

$$50 = \overset{6}{\cancel{A}}[(s+1)(2s+4) + (s^2+4s+13)] - 5(2s+4) +$$
$$+ (Cs+D)2(s+1) + (s+1)^2(C)$$

$s=-1$

$$50 = A(10) - 5(2(-1)+4)$$
$$50 = 10A - 10$$
$$60 = 10A$$
$$6 = A$$

$$\begin{array}{r} {}^{4}17 \\ 6 \\ \hline 102 \end{array}$$

-2

$$6 = A$$

$$s = -2 \quad 50 = 6(4 - 8 + 13) + (-2C + D)\overbrace{(2)(-1)}^{-2} + C$$

$$-4 = +4C - 2D + C$$

$$-4 = 5C - 2D$$

$$s = 0 \quad 50 = 6[4 + 13] - 5(4) + D(2)(1) + C$$

$$50 = 102 - 20 + 2D + C$$

$$50 = 82 + 2D + C$$

$$-32 = C + 2D$$

$$\begin{array}{rl} & -4 = 5C - 2D \\ + & -32 = C + 2D \leftarrow \\ \hline & -36 = 6C \end{array}$$

$$-6 = C$$

$$-32 = -6 + 2D$$

$$-26 = 2D$$

$$-13 = D$$

$$\mathcal{L}^{-1}\left\{ \frac{6}{s+1} - \frac{5}{(s+1)^2} - \frac{+6s+13}{s^2+4s+13} \right\}$$

No s

11. $e^{\alpha t}t^n$, $n = 1, 2, 3, \ldots$ | $\frac{n!}{(s-\alpha)^{n+1}}$

12. $e^{\alpha t}\sin\omega t$ | $\frac{\omega}{(s-\alpha)^2+\omega^2}$

13. $e^{\alpha t}\cos\omega t$ | $\frac{s-\alpha}{(s-\alpha)^2+\omega^2}$

$$\frac{6s+13}{s^2+4s+13} = \frac{6s+13}{s^2+4s+4+13-4} = \frac{6s+12+1}{(s+2)^2+3^2}$$

$$= \frac{6(s+2)}{(s+2)^2+3^2} + \frac{1}{(s+2)^2+3^2}$$

$$\mathcal{L}^{-1}\left\{\frac{6}{s+1}\right\} = 6e^{-t}$$

row #4

$$\mathcal{L}^{-1}\left\{\frac{-5}{(s+1)^2}\right\} = -5e^{-t}\,t$$ row #11

$$\mathcal{L}^{-1}\left\{\frac{6(s+2)}{(s+2)^2+3^2}\right\} = 6e^{-2t}\cos(3t)$$ row #13

$\alpha = -2$

$\omega = 3$

$$\frac{1}{3}\mathcal{L}^{-1}\left\{\frac{1(3)}{(s+2)^2+3^2}\right\} = \frac{1}{3}e^{-2t}\sin(3t)$$ row #12

$$\boxed{\mathcal{L}^{-1}\{F(s)\} = 6e^{-t} - 5te^{-t} - 6e^{-2t}\cos(3t) - \frac{1}{3}e^{-2t}\sin 3t}$$