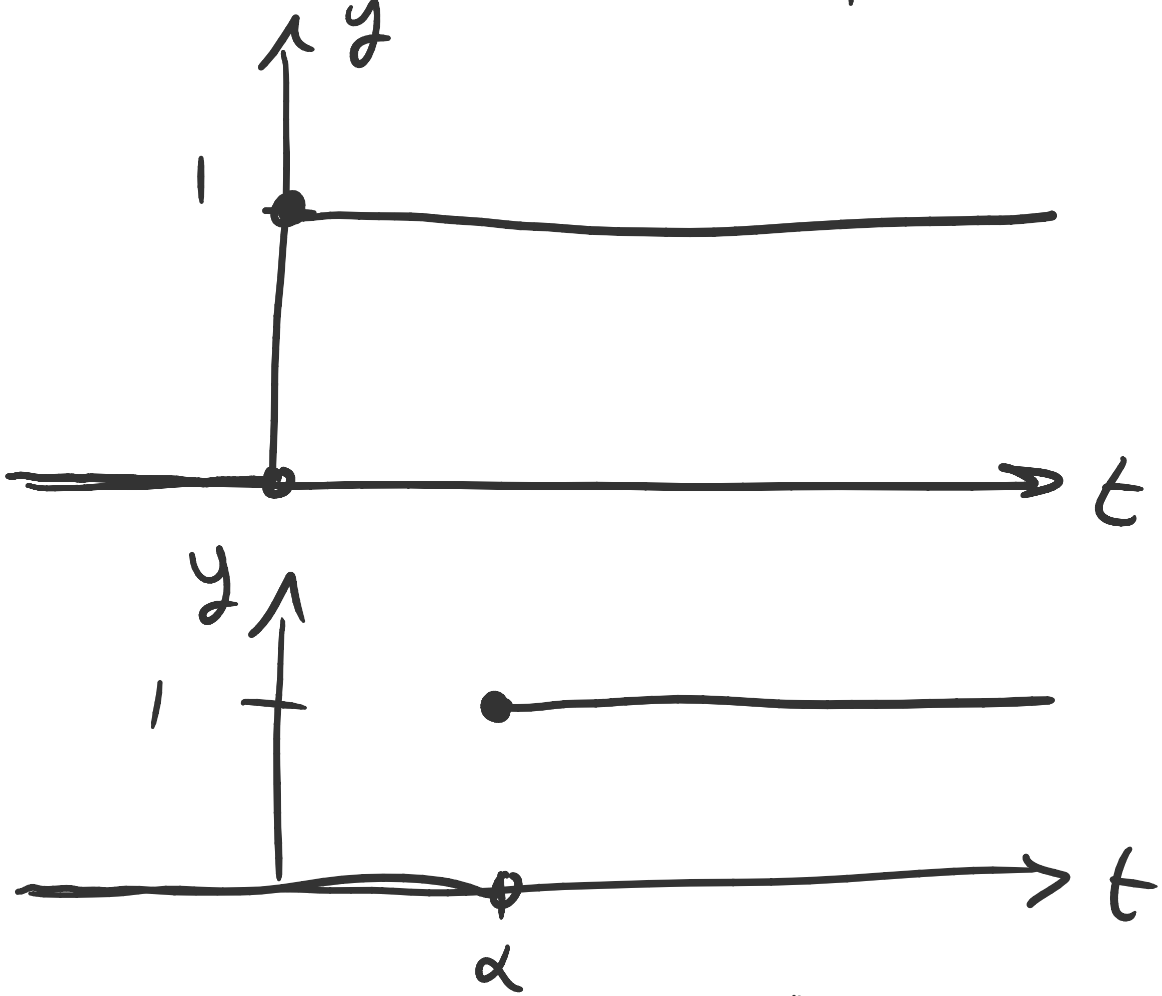
Section 5.2a Laplace Transform Pairs

Friday, April 12, 2019 11:53 AM

Heaviside step function



$$h(t) = \begin{cases} 1, & t \geq 0 \\ 0, & t < 0 \end{cases}$$

$$h(t-\alpha)$$

$$\mathcal{L}\{h(t)\} = \int_0^\infty h(t) e^{-st}\, dt = \int_0^\infty e^{-st}\, dt$$

$$\mathcal{L}\{h(t)\} = \int_0 h(t) e \, dt = \int_0 e \, dt$$

$$= -\frac{1}{s}\left[e^{-st}\right]_0^{\infty} = \frac{1}{s}, \quad s > 0$$

$$\mathcal{L}\{1\} = \frac{1}{s}, \quad s > 0$$

$$\mathcal{L}\{h(t-\alpha)\} = \int_0^{\infty} h(t-\alpha) e^{-st} \, dt$$

$$= \int_{\alpha}^{\infty} 1 e^{-st} \, dt$$

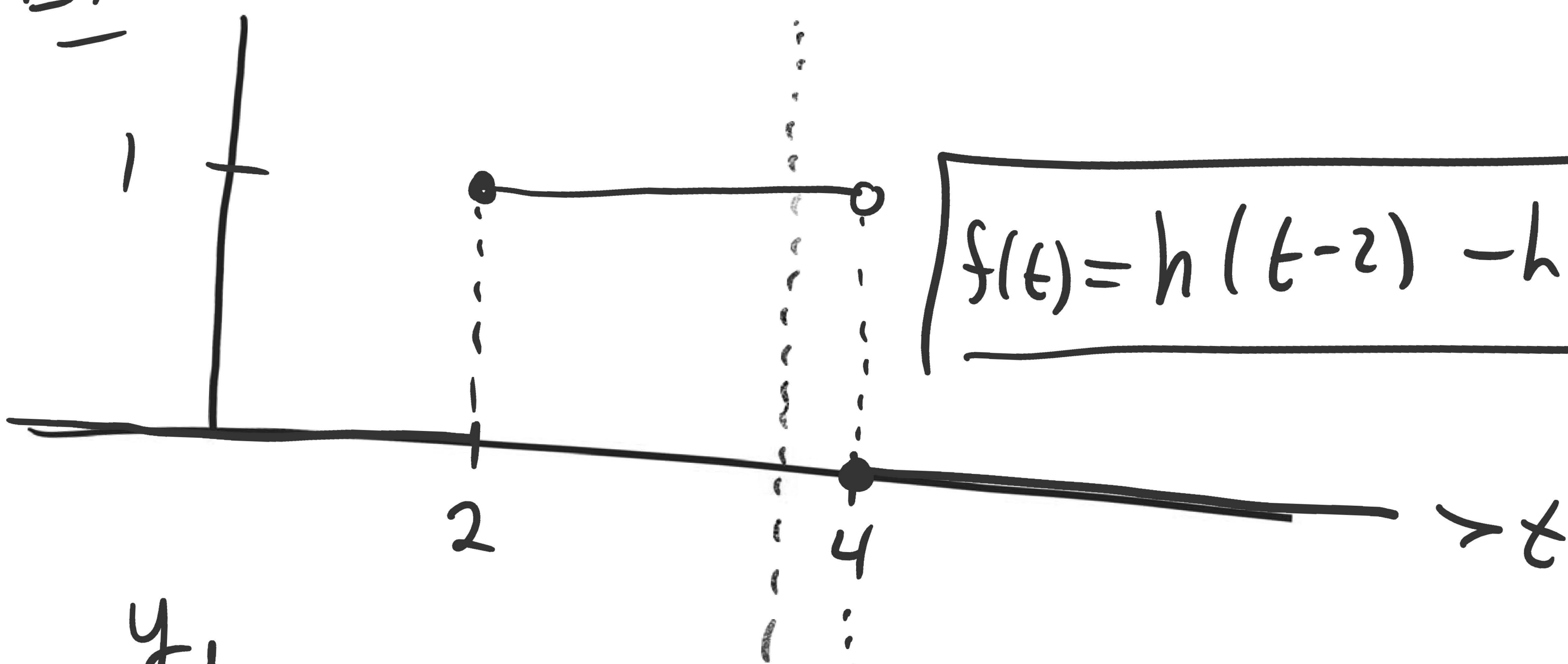
$$= -\frac{1}{s}\left[e^{-st}\right]_{\alpha}^{\infty}$$

$$= -\frac{1}{s}\left[0 - e^{-\alpha s}\right]$$

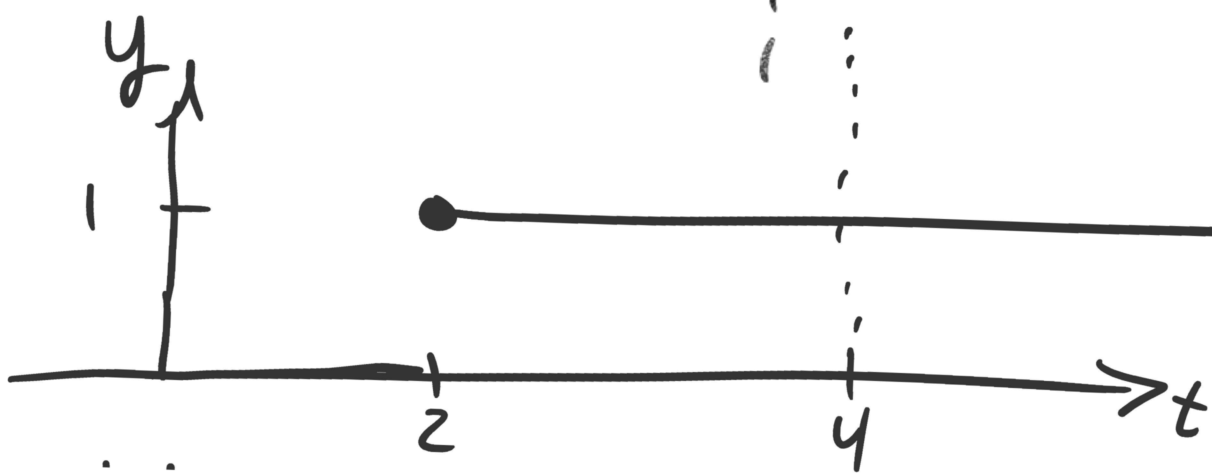
$$\boxed{\mathcal{L}\{h(t-\alpha)\} = \frac{1}{s} e^{-\alpha s}}$$

$$\boxed{\mathcal{L}\{h(t-\alpha)\} = \frac{1}{s}e^{-\alpha s}}$$

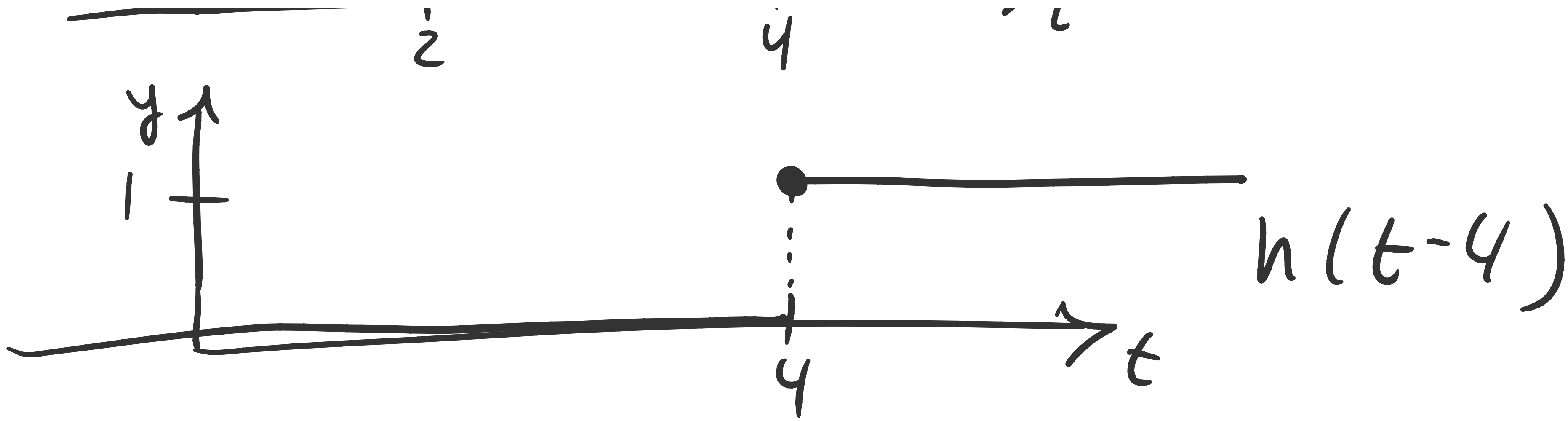
Ex:



$$\boxed{f(t) = h(t-2) - h(t-4)}$$



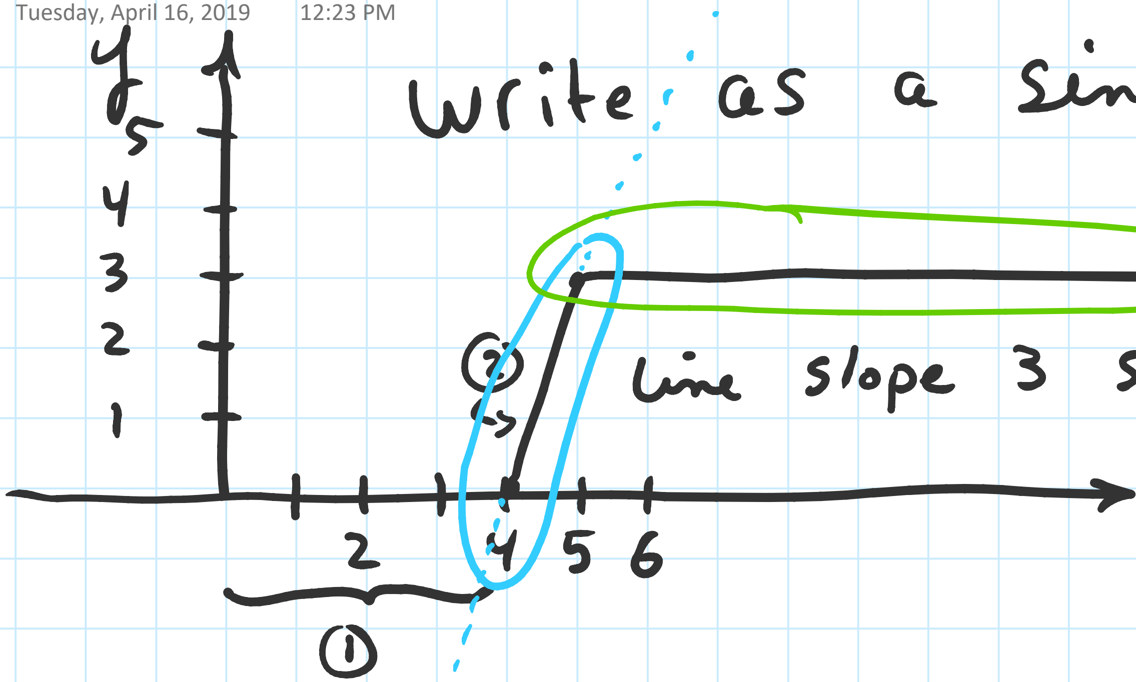
$h(t-2)$


y
1
4
t
u(t-4)

Section 5.2b Laplace Transform Pairs

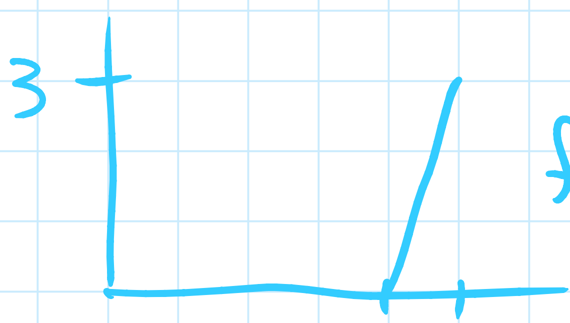
Tuesday, April 16, 2019 12:23 PM

write as a single function.

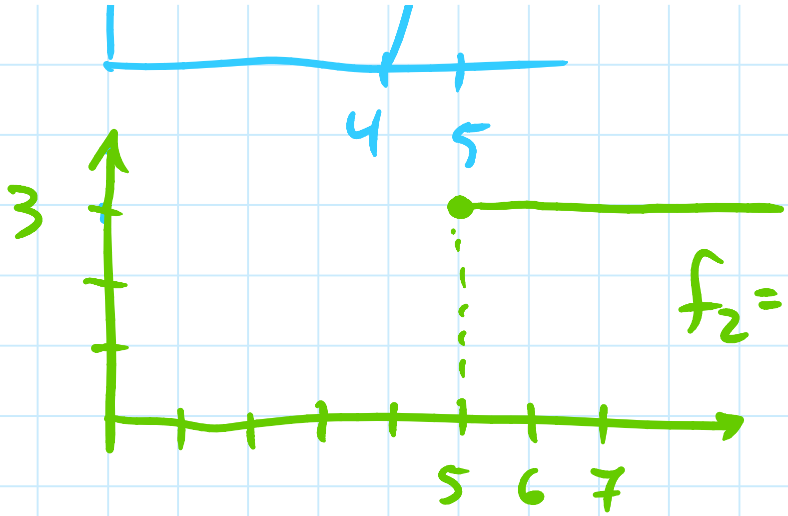


line slope 3 shifted right 4 units

$$f(t) = \begin{cases} 0 & t < 4 \\ 3(t-4) & 4 \le t < 5 \\ 3 & t \ge 5 \end{cases}$$



$$f_1 = 3(t-4)\overbrace{\left[h(t-4) - h(t-5)\right]}^{=1 \text{ between } 4 \,\&\, 5}$$



$$f_2 = 3h(t-5)$$

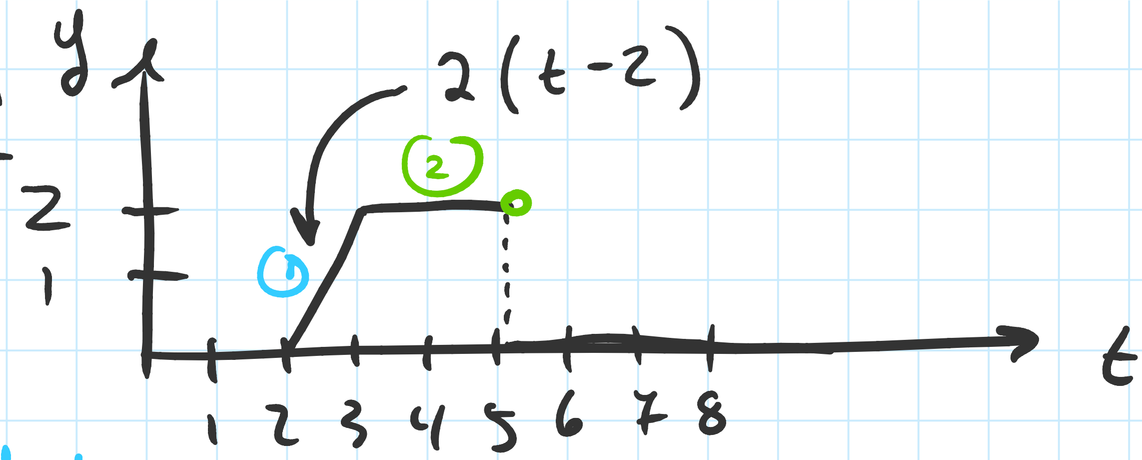
$$f(t) = f_1 + f_2 = 3(t-4)\left[h(t-4) - h(t-5)\right] + 3h(t-5)$$

$$= 3(t-4)h(t-4) - 3(t-4)h(t-5) + 3h(t-5)$$

$$= 3(t-4)h(t-4) + h(t-5)\left[\underbrace{-3(t-4)+3}_{\substack{-3t+12+3 \\ -3(t-5)}}\right]$$

$$= 3(t-4)h(t-4) + (-3(t-5))h(t-5)$$

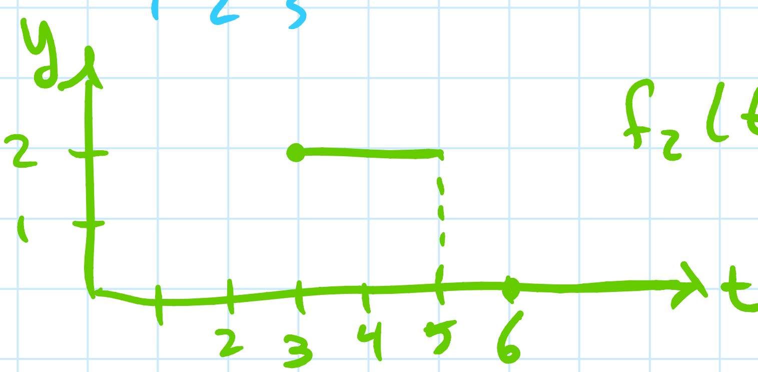
Ex :



(1)

$$f_1(t) = 2(t-2)\left[h(t-2) - h(t-3)\right]$$

(2)



$$f_2(t) = 2\left[h(t-3) - h(t-5)\right]$$

$$f(t) = 2(t-2)\,h(t-2) - 2(t-2)\,h(t-3) + 2\,h(t-3) - 2h(t-5)$$

$$= 2(t-2)\,h(t-2) + h(t-3)\left[2 - 2(t-2)\right] - 2h(t-5)$$

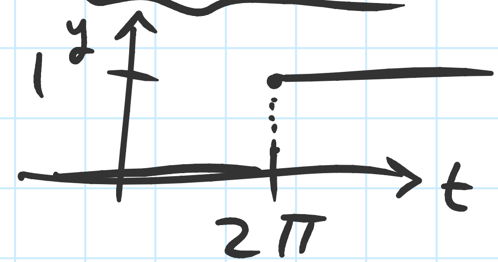
$= 2(t-2)\,h(t-2) + h(t-3)\left[2 - 2(t-2)\right] - 2h(t-5)$

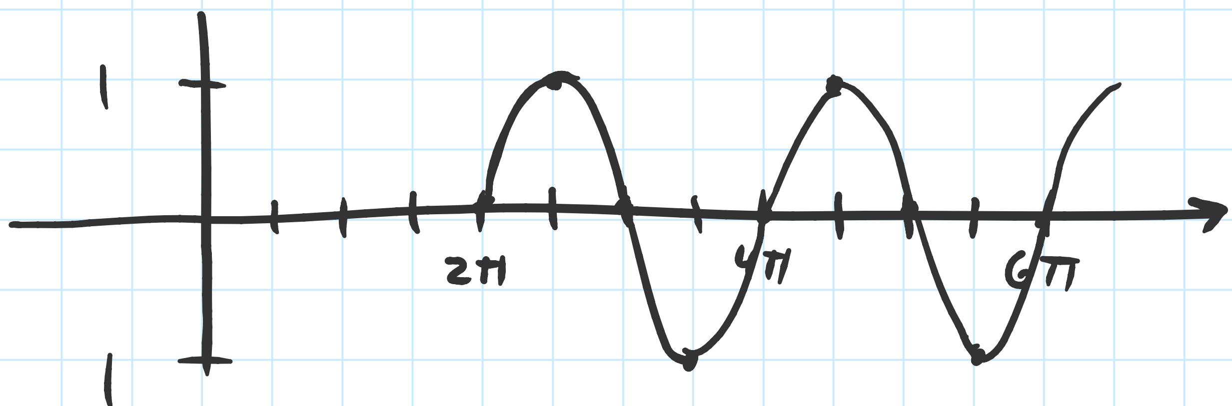
$2 - 2t + 4$

$-2(t-3)$

$$\boxed{= 2(t-2)\,h(t-2) - 2(t-3)\,h(t-3) - 2h(t-5)}$$

<u>Ex:</u> graph $f(t) = \sin(t-2\pi)\,\underbrace{h(t-2\pi)}$





Also $g(t) = \sin(t)\,h(t-2\pi)$

Shift Theorems

1. $\mathcal{L}\{e^{\alpha t}\underbrace{f(t)}\} = F(s-\alpha)$

$\mathcal{L}\{f(t)\} = F(s)$ #9 on table

same

2. $\mathcal{L}\{f(t-\alpha)\,h(t-\alpha)\} = e^{-\alpha s}F(s)$

Ex: $\mathcal{L}\{\sin(t-2\pi)\,h(t-2\pi)\} = \boxed{e^{-2\pi s}\left(\frac{1}{s^2+1}\right)}$

$f(t) = \sin(t)$

$F(s) = \mathcal{L}\{\sin t\} = \frac{1}{s^2+1}$

$F(s) = \mathcal{L}\{\sin t\} = \quad s^2+1$

Ex: $\mathcal{L}\{e^{2t}\cos(3t)\} = \boxed{\dfrac{s-2}{(s-2)^2+9}}$

$f(t) = \cos(3t)$

$F(s) = \mathcal{L}\{\cos 3t\} = \dfrac{s}{s^2+3^2} \leftarrow$ row 6

Also row 13

Ex: $\mathcal{L}\{3t^2 + 2t + 1\} = 3\mathcal{L}\{t^2\} + 2\mathcal{L}\{t\} + \mathcal{L}\{1\}$

$\boxed{= 3\left(\dfrac{2}{s^3}\right) + 2\left(\dfrac{1}{s^2}\right) + \dfrac{1}{s}}$

row 3

$0! = 1$

$\underline{Ex}$: $\mathcal{L}\{e^{4t}(3t^2+2t+1)\}$ $\quad \alpha = 4$ in Shift th$\underline{m}$ 1

$$\boxed{= 3\,\frac{2}{(s-4)^3} + 2\left(\frac{1}{(s-4)^2}\right) + \frac{1}{s-4}}$$

Ex: $\mathcal{L}\{e^{3t-3}\, h(t-1)\}$

Shift thm 2

$\mathcal{L}\{f(t-\alpha)\, h(t-\alpha)\} = e^{-\alpha s} F(s)$

$= \mathcal{L}\{e^{3(t-1)}\, h(t-1)\}$

$f(t) = e^{3t}$ $\qquad f(t-1) = e^{3(t-1)}$

$F(s) = \frac{1}{s-3}$ $\qquad$ row 4 $\qquad \alpha = 1$

$$\boxed{\mathcal{L}\{e^{3(t-1)}\, h(t-1)\} = e^{-s}\left(\frac{1}{s-3}\right)}$$

Section 5.2c Laplace Transform Pairs

Friday, April 19, 2019 8:53 AM

Inverse Laplace

Ex: $F(s) = \frac{3}{s} + \frac{24}{s^2}$

$$f(t) = \mathcal{L}^{-1}\{F(s)\} = 3\mathcal{L}^{-1}\left\{\frac{1}{s}\right\} + 24\mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\}$$

$$= 3(1) + 24(t) = \boxed{3 + 24t}$$

Ex: $F(s) = \frac{2s-4}{(s-2)^2+9}$

$$f(t) = \mathcal{L}^{-1}\left\{\frac{2s-4}{(s-2)^2+9}\right\} = 2\mathcal{L}^{-1}\left\{\frac{(s-2)}{(s-2)^2+3^2}\right\}$$

$$\boxed{= 2e^{2t}\cos(3t)}$$

$$\boxed{= 2e^{2t}\cos(3t)}$$

$$\mathcal{L}^{-1}\left\{\frac{s-\alpha}{(s-\alpha)^2+\omega^2}\right\} = e^{\alpha t}\cos(\omega t)$$

Ex: $F(s) = e^{-2s}\dfrac{3}{s^2+9}$

$$\mathcal{L}^{-1}\left\{\underbrace{e^{-2s}}_{\text{shift}}\left(\frac{3}{s^2+9}\right)\right\} = \underbrace{f(t-2)}\,h(t-2)$$

$\uparrow F(s)$

$$\boxed{= \sin(3(t-2))\,h(t-2)}$$

$$f(t) = \mathcal{L}^{-1}\left\{\frac{3}{s^2+9}\right\} = \sin(3t)$$

shift thm $\mathcal{L}\{f(t-\alpha)\,h(t-\alpha)\} = e^{-\alpha s}F(s)$

Ex: $F(s) = \dfrac{4s-6}{s^2-2s+10} = \dfrac{4s-4-2}{(s-1)^2+9}$

Ex: $F(s) = \dfrac{\quad}{s^2 - 2s + 10} = (s-1)^2 + 9$

complete the square on the denominator

$$s^2 - 2s \underline{+1} + 10 \underline{-1} = (s-1)^2 + 9$$

$$F(s) = \frac{4(s-1) - 2}{(s-1)^2 + 9} = \underbrace{\frac{4(s-1)}{(s-1)^2 + 3^2}}_{\text{row } \#13} - \underbrace{\frac{2}{(s-1)^2 + 3^2}}_{\text{row } \#12}$$

$$f(t) = 4\mathcal{L}^{-1}\left\{\frac{s-1}{(s-1)^2 + 3^2}\right\} - \frac{2}{3}\mathcal{L}^{-1}\left\{\frac{1(3)}{(s-1)^2 + \underline{3}^2}\right\}$$

$$\boxed{= 4e^{1t}\cos(3t) - \frac{2}{3}e^{t}\sin(3t)}$$

Laplace Transforms of derivatives & $\int$.

$$\mathcal{L}\{f'(t)\} = s\,\mathcal{L}\{f(t)\} - f(0)$$
$$= s\,F(s) - f(0)$$

$$\mathcal{L}\{f''(t)\} = s^2 F(s) - s f(0) - f'(0)$$

$$\mathcal{L}\left\{\int_0^t f(u)\,du\right\} = \frac{\mathcal{L}\{f(t)\}}{s} = \frac{F(s)}{s}$$

$y(0)=0$

Ex: $$\frac{dy}{dt} + 6y(t) + 9\int_0^t y(\tau)\,d\tau = 1$$

$$\mathcal{L}\{y'\} + 6\,\mathcal{L}\{y\} + 9\,\mathcal{L}\left\{\int_0^t y(\tau)\,d\tau\right\} = \mathcal{L}\{1\}$$

$$s\underline{Y}(s) + \cancel{y(0)}^{\,0} + 6\,Y(s) + 9\,\frac{Y(s)}{s} = \frac{1}{s}$$

$$Y(s)\left[s + 6 + \frac{9}{s}\right] = \frac{1}{s}$$

$$Y(s) = \frac{1}{s\left(s+6+\frac{9}{s}\right)}$$

$$Y(s) = \frac{1}{s^2+6s+9} = \frac{1}{(s+3)^2}$$

$$\mathcal{L}^{-1}\{Y(s)\} = y(t) = \mathcal{L}^{-1}\left\{\frac{1}{(s+3)^2}\right\}$$

$n = 1$

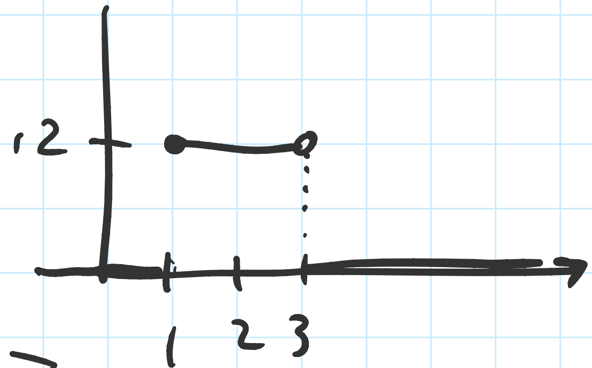
$\alpha = -3$

$$\boxed{y(t) = e^{-3t}\,t}$$

row 11 $\mathcal{L}^{-1}\left\{\frac{n!}{(s-\alpha)^{n+1}}\right\} = e^{\alpha t}\,t^n$

Ex: $y' + 4y = g(t)$ $\quad y(0) = 0$

$$g(t) = \begin{cases} 0 & 0 \le t < 1 \\ 12 & 1 \le t < 3 \\ 0 & 3 \le t < \infty \end{cases}$$



$$g(t) = 12\left[h(t-1) - h(t-3)\right]$$

row 15 $\quad \mathcal{L}\{h(t-\alpha)\} = \dfrac{e^{-\alpha s}}{s}$

$$\mathcal{L}\{y'\} + 4\mathcal{L}\{y\} = 12\mathcal{L}\{h(t-1)\} - 12\mathcal{L}\{h(t-3)\}$$

$$sY(s) + \cancelto{0}{y(0)} + 4Y(s) = 12\frac{e^{-s}}{s} - 12\frac{e^{-3s}}{s}$$

$$Y(s)(s+4) = \frac{12}{s}\left(e^{-s} - e^{-3s}\right)$$

$$Y(s)(s+4) = \frac{12}{s}(e^{-s} - e^{-3s})$$

$$Y(s) = \frac{12}{s(s+4)}(e^{-s} - e^{-3s})$$

$$\left[\frac{12}{s(s+4)} = \frac{A}{s} + \frac{B}{s+4}\right] s(s+4)$$

$$12 = A(s+4) + B(s)$$

$s=0 \quad 4A=12 \Rightarrow A = 3$

$s=-4 \quad -4B=12 \Rightarrow B = -3$

$$\frac{12}{s(s+4)} = \frac{3}{s} - \frac{3}{s+4}$$

$$\mathcal{L}^{-1}\{Y(s)\} = \left(\frac{3}{s} - \frac{3}{s+4}\right)\left[e^{-s} - e^{-3s}\right]$$

$$\mathcal{L}^{-1}\{Y(s)\} = \left(\frac{3}{s} - \frac{3}{s+4}\right)\left[e^{-s} - e^{-3s}\right]$$

$$= \mathcal{L}^{-1}\left\{\underbrace{\left(\frac{3}{s}-\frac{3}{s+4}\right)}_{F(s)} e^{-s} - \underbrace{\left(\frac{3}{s}-\frac{3}{s+4}\right)}_{F(s)} e^{-3s}\right\}$$

$$\mathcal{L}^{-1}\left\{\underbrace{\frac{3}{s}}_{\text{row } 1} - \underbrace{\frac{3}{s+4}}_{\text{row } 4}\right\} = 3 - 3e^{-4t} = f(t)$$

$$\boxed{y(t) = \left(3-3e^{-4(t-1)}\right)h(t-1) - \left(3-3e^{-4(t-3)}\right)h(t-3)}$$