Contents

5.1 Introduction to Laplace Transforms

5.1.1 The Laplace Transform

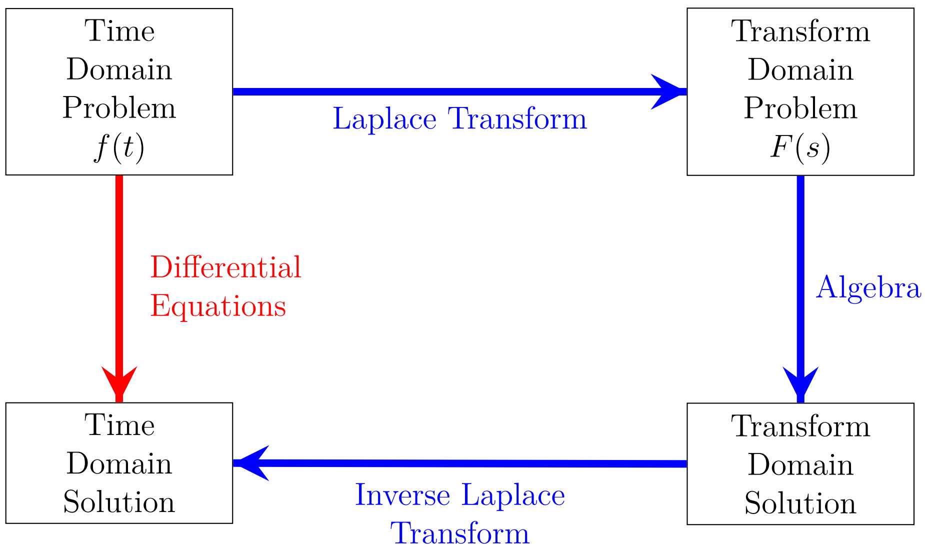
When we are looking at problems involving Laplace Transforms there are always 2 domains to which you need to pay attention.

1. The time domain: variable (t)

2. The transform domain: variable (s)

When you work problems you must be either entirely in the time domain or the the transform domain. As a general rule you DO NOT work in both domains at the same time.

This flow chart shows the two different ways to solve a differential equation. You can either take the direct route or you can take the Laplace Transform route.



All Laplace transform problems have 3 steps.

1. Laplace transform to make the problem algebraic (simpler).

2. Solve/simplify the transformed problem with algebra.

3. Inverse Laplace transform to find solution.

Definition of Laplace Transform

Definition 5.1. Let $f(t)$ be defined for $t \geq 0$ and let s be a real number. Then the **Laplace transform** of $f(t)$, denoted $\mathcal{L}\{f(t)\}$, is the function $F(s)$ defined by

$$\mathcal{L}\{f(t)\} = F(s) = \int_0^\infty e^{-st} f(t)\, dt, \tag{5.1}$$

for those values of s for which the improper integral converges.

Example 5.1.1. Find the Laplace transform of $f(t) = 8$

Solution: Recall the definition of an improper integral. If g is integrable over the interval $[a, T]$ for every $T > a$, then the improper integral of g over $[a, \infty)$ is defined as

$$\int_a^\infty g(t)\, dt = \lim_{T\to\infty} \int_a^T g(t)\, dt. \tag{5.2}$$

We say that the improper integral converges if the limit in (5.2) exists; otherwise, we say that the improper integral diverges or does not exist. So when we integrate for a Laplace transform we use the limit definition of the improper integral.

$$\begin{aligned}
\mathcal{L}\{8\} &= \int_0^\infty 8e^{-st}\, dt \\
&= \lim_{T\to\infty} \left(\frac{-1}{s}\right) \left[8e^{-st}\right]_0^T \\
&= \lim_{T\to\infty} \left(\frac{-1}{s}\right) \left[8e^{-sT} - 8\right] \\
&= \begin{cases} \dfrac{8}{s} & \text{for } s > 0 \\ \infty & \text{for } s < 0 \end{cases}
\end{aligned}$$

Notice that if $s < 0$ then e^{-sT} has a positive exponent and $\lim_{T\to\infty} e^{-sT} = \infty$
So we have found the Laplace transform of $f(t) = 8$:

$$\boxed{\mathcal{L}\{8\} = \frac{8}{s} \qquad \text{for } s > 0}$$

Note: Rather than using the limit for the improper integral we will realize that we are taking the limit but we will instead simply write

$$\int_0^\infty 8e^{-st}\, dt = \left(\frac{-1}{s}\right) \left[8e^{-st}\right]_0^\infty$$

Example 5.1.2. Find the Laplace transform of $f(t) = t^2$

Solution: This integral will require integration by parts twice. We can also look it up on a table of integrals:

$$\int t^n e^{at}\,dt = \frac{t^n e^{at}}{a} - \frac{n}{a}\int t^{n-1}e^{at}\,dt$$

$$\begin{aligned}
\mathcal{L}\left\{t^2\right\} &= \int_0^\infty t^2 e^{-st}\,dt \\
&= \left[-\frac{t^2 e^{-st}}{s} - \frac{2te^{-st}}{s^2} - \frac{2e^{-st}}{s^3}\right]_0^\infty \\
&= \left[\left(-\frac{t^2}{s} - \frac{2t}{s^2} - \frac{2}{s^3}\right)e^{-s(\infty)} + \frac{2}{s^3}\right] \\
&= \begin{cases} \dfrac{2}{s^3} & \text{for } s > 0 \\ \infty & \text{for } s < 0 \end{cases}
\end{aligned}$$

$$\boxed{\mathcal{L}\left\{t^2\right\} = \frac{2}{s^3}, \qquad \text{for } s > 0}$$

Example 5.1.3. Find the Laplace transform of $f(t) = e^{at}$

Example 5.1.4. Find the Laplace transform of $f(t) = e^{t^2}$

$$\begin{aligned}\mathcal{L}\left\{e^{t^2}\right\} &= \int_0^\infty e^{t^2} e^{-st}\, dt \\ &= \int_0^\infty e^{t^2 - st}\, dt \\ &= \int_0^\infty e^{t(t-s)}\, dt\end{aligned}$$

This integral will always diverge because,for any $s > 0$ the exponent $t(t-s)$ is positive for $t > s$. This means that the Laplace Transform of $f(t) = e^{t^2}$ DOES NOT EXIST.

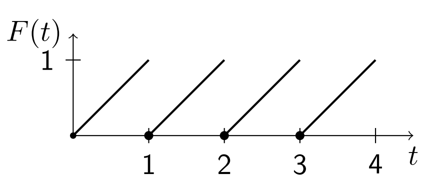
5.1.2 Existence of the Laplace Transform

For a Laplace transform of a function $f(t)$ to exist two things are required:

1. The function must be **piecewise continuous**:
   - Has a finite number of discontinuities on every finite interval.
   - The limit from the left and the limit from the right exist at all discontinuities.
2. The function must be **exponentially bounded**: There are constants M and a with $M \geq 0$, such that
$$|f(t)| < Me^{at}, \qquad 0 \leq t < \infty$$

Example 5.1.5. The periodic saw tooth wave satisfies these conditions:

$$f(t) = t, \qquad 0 \leq t < 1, \qquad f(t+1) = f(t)$$



Laplace Rules:

1. The laplace transform is a linear operator: Suppose $f(t) = C_1 f_1(t) + C_2 f_2(t)$
$$\mathcal{L}\{f(t)\} = C_1 \mathcal{L}\{f_1(t)\} + C_2 \mathcal{L}\{f_2(t)\}$$
2. If $f(t) = f_1(t) f_2(t)$ then $\mathcal{L}\{f(t)\}$ exists.

5.1.3 The Inverse Laplace Transform and Uniqueness

Example 5.1.6. Consider the two functions:

$$f_1(t) = e^{-t} \qquad\qquad f_2(t) = \begin{cases} e^{-t} & \text{for } t \notin \mathbb{Z} \\ 0 & \text{for } t \in \mathbb{Z} \end{cases}$$

They both have the same Laplace transform: $\mathcal{L}\left\{f_1(t)\right\} = \mathcal{L}\left\{f_2(t)\right\} = \dfrac{1}{s+1}$

So if we want to talk about the inverse Laplace transform of $\dfrac{1}{s+1}$ we want to know what function has Laplace transform $\dfrac{1}{s+1}$. As we can see there can be more than one answer but, unless there is a compelling reason to do otherwise, we will always assume that the inverse Laplace transform is the continuous answer and we would say:

$$\mathcal{L}^{-1}\left\{\frac{1}{s+1}\right\} = e^{-t}$$

Once we know the Laplace transform we can put it in a table. See the text or the end of these notes for an incomplete list of Laplace transform pairs. If you look at row 4 of the table 1 you find that

$$\mathcal{L}\left\{e^{\alpha t}\right\} = \frac{1}{s-\alpha} \qquad \text{and} \qquad \mathcal{L}^{-1}\left\{\frac{1}{s-\alpha}\right\} = e^{\alpha t}$$

Example 5.1.7. Find the inverse Laplace transform of

$$F(s) = \frac{3}{s+2} + \frac{5}{s-2}$$

Solution:

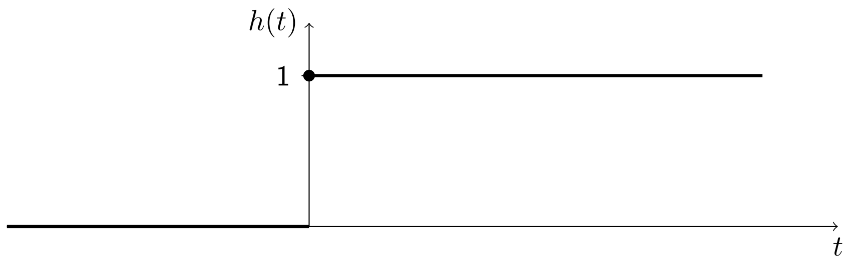
$$\begin{aligned} f(t) &= \mathcal{L}^{-1}\left\{F(s)\right\} \\ &= \mathcal{L}^{-1}\left\{\frac{3}{s+2}\right\} + \mathcal{L}^{-1}\left\{\frac{5}{s-2}\right\} \\ &= 3\mathcal{L}^{-1}\left\{\frac{1}{s+2}\right\} + 5\mathcal{L}^{-1}\left\{\frac{1}{s-2}\right\} \\ &= 3e^{-2t} + 5e^{2t} \end{aligned}$$

So $\boxed{f(t) = 3e^{-2t} + 5e^{2t}}$

5.2 Laplace Transform Pairs

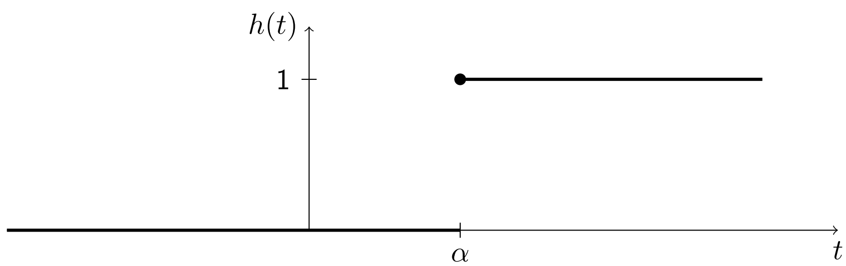
The unit step function or Heaviside step function is defined as follows:

$$h(t) = \begin{cases} 1 & t \geq 0 \\ 0 & t < 0 \end{cases}$$



Shifted by α

$$h(t-\alpha) = \begin{cases} 1 & t \geq \alpha \\ 0 & t < \alpha \end{cases}$$



Example 5.2.1. Find the Laplace transform of $h(t)$ and $h(t-\alpha)$.

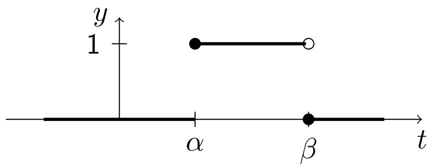
Solution:

$$\begin{aligned} \mathcal{L}\{h(t)\} &= \int_0^\infty h(t)e^{-st}\,dt = \int_0^\infty e^{-st}\,dt = -\frac{1}{s}\left[e^{-st}\right]_0^\infty \\ \mathcal{L}\{h(t)\} &= \frac{1}{s}, \quad s > 0 \end{aligned}$$

$$\begin{aligned} \mathcal{L}\{h(t-\alpha)\} &= \int_0^\infty h(t-\alpha)e^{-st}\,dt = \int_\alpha^\infty (1)e^{-st}\,dt = -\frac{1}{s}\left[e^{-st}\right]_\alpha^\infty \\ &= -\frac{1}{s}\left[0 - e^{-\alpha s}\right] \\ \mathcal{L}\{h(t-\alpha)\} &= \frac{1}{s}e^{-\alpha s}, \quad s > \alpha \end{aligned}$$

What is $\mathcal{L}(1)$?

Example 5.2.2. Write the following graph as a single function.



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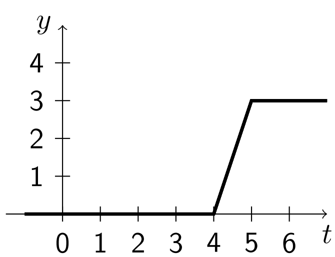

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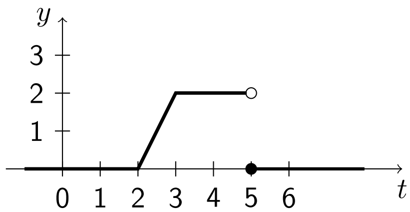


Example 5.2.5. Graph the function $f(t) = \sin(t - 2\pi)h(t - 2\pi)$

5.2.1 Shift Theorems

1. $\mathcal{L}\{e^{\alpha t}f(t)\} = F(s-\alpha)$

   where $\mathcal{L}\{f(t)\} = F(t)$ See #9 on table 1

2. $\mathcal{L}\{f(t-\alpha)h(t-\alpha)\} = e^{-\alpha s}F(s)$

Example 5.2.6. Find the Laplace transform of $f(t) = \sin(t-2\pi)h(t-2\pi)$

Example 5.2.7. Find the Laplace transform of $f(t) = e^{2t}\cos(3t)$

Example 5.2.8. Find the Laplace transform of $f(t) = 3t^2 + 2t + 1$

Example 5.2.9. Find the Laplace transform of $f(t) = e^{4t}(3t^2 + 2t + 1)$

5.2.2 Inverse transforms

Example 5.2.10. Find the Laplace transform of $f(t) = e^{3t-3}h(t-1)$

Example 5.2.11. Find the inverse Laplace transform of $F(s) = \dfrac{3}{s} + \dfrac{24}{s^2}$

Example 5.2.12. Find the inverse Laplace transform of $F(s) = \dfrac{2s-4}{(s-2)^2+9}$

Example 5.2.13. Find the inverse Laplace transform of $F(s) = \dfrac{4s-6}{s^2-2s+10}$

5.2.3 Laplace transforms of derivatives

1. $\mathcal{L}\{f'(t)\} = s\mathcal{L}\{f(t)\} - f(0) = sF(s) - f(0)$

2. $\mathcal{L}\{f''(t)\} = s^2F(s) - sf(0) - f'(0)$

3. $\mathcal{L}\left\{\int_0^t f(u)\,du\right\} = \dfrac{\mathcal{L}\{f(t)\}}{s} = \dfrac{F(s)}{s}$

Example 5.2.14. Find the Laplace transform of the differential equation

$$\frac{dy}{dt} + 6y(t) + 9\int_0^t y(\tau)\,d\tau = 1$$

and solve for $Y(s)$. Then use the inverse Laplace transform to find the solution to the equation $y(t)$

Example 5.2.15. Solve the differential equation

$$y' + 4y = g(t), \qquad y(0) = 0$$

where

$$g(t) = \begin{cases} 0, & 0 \leq t < 1 \\ 12, & 1 \leq t < 3 \\ 0, & 3 \leq t < \infty \end{cases}$$

5.3 The Method of Partial Fractions

	Denominator	Partial Fraction
Linear	$(s+a)$	$\dfrac{A}{s+a}$
Repeated Linear	$(s+a)^n$	$\dfrac{A_1}{s+a}+\dfrac{A_2}{(s+a)^2}+\cdots+\dfrac{A_n}{(s+a)^n}$
Quadratic	s^2+as+b	$\dfrac{A_1s+A_2}{s^2+as+b}$
Repeated Quadratic	$(s^2+as+b)^2$	$\dfrac{A_1s+A_2}{s^2+as+b}+\dfrac{A_3s+A_4}{(s^2+as+b)^2}$

Example 5.3.1. Denominator is a product of distinct linear factors.

$$\frac{3x+7}{x^2+6x+5}$$

Example 5.3.2. Denominator is a product of linear factors, some of which are repeated.

$$\frac{3x^2-8x+13}{(x+3)(x-1)^2}$$

Example 5.3.3. Denominator contains irreducible quadratic factors, none of which is repeated.

$$\frac{2x^2 + x - 8}{x^3 + 4x}$$

Example 5.3.4. Find $\mathcal{L}^{-1}\left\{\dfrac{s^2+4}{s^4-s^2}\right\}$

Example 5.3.5. Find the inverse Laplace transform of

$$F(s) = \frac{50s}{(s+1)^2(s^2+4s+13)}$$

Note:
$\mathcal{L}\left\{e^{\alpha t}t^n\right\} = \frac{n!}{(s-\alpha)^{n+1}}$
$\mathcal{L}\left\{e^{\alpha t}\sin\omega t\right\} = \frac{\omega}{(s-\alpha)^2+\omega^2}$
$\mathcal{L}\left\{e^{\alpha t}\cos\omega t\right\} = \frac{s-\alpha}{(s-\alpha)^2+\omega^2}$

5.4 Laplace Transforms of Periodic Functions and System Transfer Functions

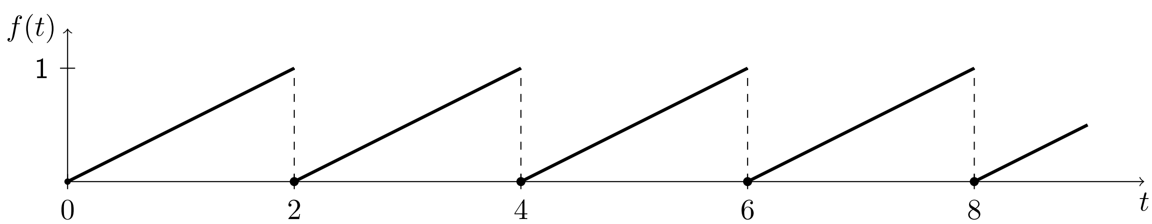
5.4.1 Periodic Functions

Laplace Transform of a Periodic Function

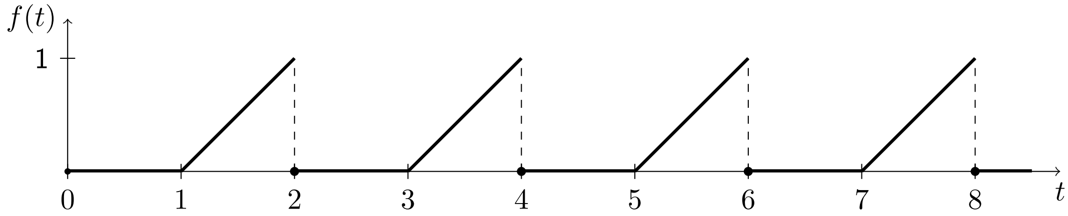
Definition 5.2. Let $f(t)$ be a piecewise continuous periodic function defined on $0 \leq t < \infty$ with period T. Then the Laplace transform of $f(t)$ is the function defined by

$$\mathcal{L}\{f(t)\} = \frac{\int_0^T e^{-st} f(t)\,dt}{1 - e^{-sT}}, \qquad s > 0 \tag{5.3}$$

Example 5.4.1. Find the Laplace transform of the function whose graph is shown



Example 5.4.2. Find the Laplace transform of the function whose graph is shown



5.4.2 System Transfer Functions

Consider the spring mass damper system we studied in Chapter 3. Given a mass m, a damper with damping coefficient γ and a spring with spring constant k then the system from equilibrium can be modeled by the differential equation

$$my'' + \gamma y' + ky = f(t), \quad t \geq 0, \qquad y(0) = 0 \quad y'(0) = 0$$

where $f(t)$ is some external force.

We can solve this using the techniques from chapter 3 or we can solve it using Laplace transforms by using

$\mathcal{L}\{g'(t)\} = sG(s) - g(0)$

$\mathcal{L}\{g''(t)\} = s^2G(s) - sg(0) - g'(0)$

$$\begin{aligned} \mathcal{L}\{my'' + \gamma y' + ky\} &= \mathcal{L}\{f(t)\} \\ m(s^2Y(s)) + \gamma(sY(s)) + kY(s) &= F(s) \\ Y(s) &= \frac{1}{ms^2 + \gamma s + k}F(s) \end{aligned}$$

We can think of $\frac{1}{ms^2 + \gamma s + k}$ as a single item and call it $\Phi(s) = \frac{1}{ms^2 + \gamma s + k}$. Then our equation in the transform domain is simply

$$Y(s) = \Phi(s)F(s)$$

where $\Phi(s)$ can be found from knowing the input $F(s)$ and the output $Y(s)$ and nothing else.

$$\Phi(s) = \frac{Y(s)}{F(s)} \tag{5.4}$$

$\Phi(s)$ is called the **system transfer function**.

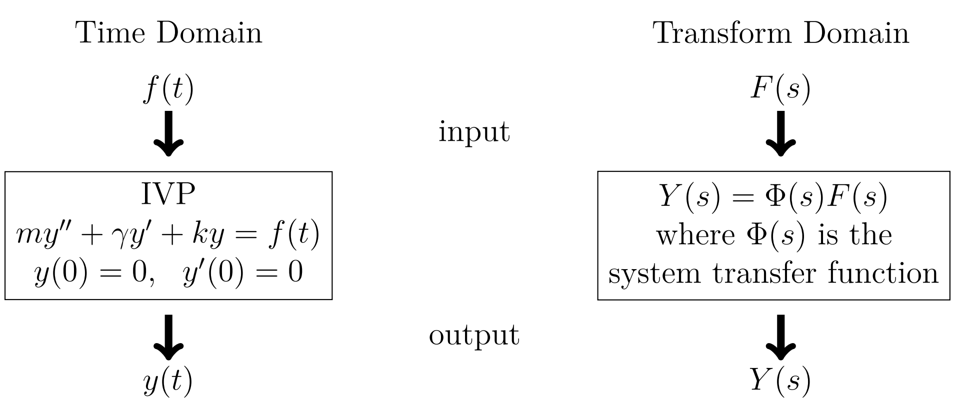


Figure 1: Two ways to solve a mechanical system. In the time domain or the Laplace domain

Example 5.4.3. Suppose we know we have the spring-mass-damper system

$$my'' + \gamma y' + ky = f(t) \quad y(0) = 0, \quad y'(0) = 0.$$

If we apply the Heaviside step function as the input forcing function $f(t) = h(t)$ then the output is $y(t) = \frac{1}{2} - \frac{1}{2}e^{-t}\cos t - \frac{1}{2}e^{-t}\sin t$. Given a new input of $\hat{f}(t) = e^{-2t}$, what is the new output $\hat{y}(t)$

5.6 Convolution

5.7 The Delta Function and Impulse Response

Table 1: **A Brief Table Of Laplace Transform Pairs**

$f(t)$, $t \geq 0$	Laplace Transform $F(s)$	
1. $h(t) = \begin{cases} 1 & t \geq 0 \\ 0, & t < 0 \end{cases}$	$\frac{1}{s}$	$(s > 0)$
2. 1	$\frac{1}{s}$	$(s > 0)$
3. $t^n, \quad n = 1, 2, 3, \ldots$	$\frac{n!}{s^{n+1}}$	$(s > 0)$
4. $e^{\alpha t}$	$\frac{1}{s - \alpha}$	$(s > \alpha)$
5. $\sin \omega t$	$\frac{\omega}{s^2 + \omega^2}$	$(s > 0)$
6. $\cos \omega t$	$\frac{s}{s^2 + \omega^2}$	$(s > 0)$
7. $\sinh bt = \frac{e^{bt} - e^{-bt}}{2}$	$\frac{b}{s^2 - b^2}$	$(s > \lvert b \rvert)$
8. $\cosh bt = \frac{e^{bt} + e^{-bt}}{2}$	$\frac{s}{s^2 - b^2}$	$(s > \lvert b \rvert)$
9. $e^{\alpha t} f(t)$, with $\lvert f(t) \rvert \leq M e^{at}$	$F(s - \alpha)$	$s > \alpha + a$
(10) - (13) are four special cases of (9)		
10. $e^{\alpha t} h(t)$	$\frac{1}{s - \alpha}$	$(s > \alpha)$
11. $e^{\alpha t} t^n, \quad n = 1, 2, 3, \ldots$	$\frac{n!}{(s - \alpha)^{n+1}}$	$(s > \alpha)$
12. $e^{\alpha t} \sin \omega t$	$\frac{\omega}{(s - \alpha)^2 + \omega^2}$	$(s > \alpha)$
13. $e^{\alpha t} \cos \omega t$	$\frac{s - \alpha}{(s - \alpha)^2 + \omega^2}$	$(s > \alpha)$
14. $f(t - \alpha) h(t - \alpha), \quad (\alpha \geq 0)$, with $\lvert f(t) \rvert \leq M e^{at}$	$e^{-\alpha s} F(s)$	$(s > a)$
15. $h(t - \alpha), \quad \alpha \geq 0$	$\frac{e^{-\alpha s}}{s}$	$(s > 0)$
16. $f'(t)$, with $f(t)$ continuous with $\lvert f'(t) \rvert \leq M e^{at}$	$sF(s) - f(0)$	$s > \max\{a, 0\}$

17. $f''(t)$, with $f'(t)$ continuous with $\lvert f''(t)\rvert \le Me^{at}$	$s^2F(s) - sf(0) - f'(0)$	$s > \max\{a, 0\}$
18. $f^{(n)}(t)$, with $f^{(n-1)}(t)$ continuous with $\lvert f^{(n)}(t)\rvert \le Me^{at}$	$s^nF(s) - s^{n-1}f(0) - \cdots$ $-sf^{(n-2)} - f^{(n-1)}(0)$	$s > \max\{a, 0\}$
19. $\int_0^t f(u)\,du$, with $\lvert f^{(n)}(t)\rvert \le Me^{at}$	$\frac{F(s)}{s}$	$s > \max\{a, 0\}$
20. $\int_0^t f(t-\lambda)g(\lambda)\,d\lambda$	$F(s) \cdot G(s)$	
21. $t\cos\omega t$	$\frac{s^2-\omega^2}{(s^2+\omega^2)^2}$	$(s > 0)$
22. $t\sin\omega t$	$\frac{2\omega s}{(s^2+\omega^2)^2}$	$(s > 0)$
23. $\sin\omega t - \omega t\cos\omega t$	$\frac{2\omega^3}{(s^2+\omega^2)^2}$	$(s > 0)$
24. $\omega t - \sin\omega t$	$\frac{\omega^3}{s^2(s^2+\omega^2)^2}$	$(s > 0)$
25. $tf(t)$	$-F'(s)$	
26. $t^kf(t)$	$(-1)^kF^{(k)}(s)$	
27. $\delta(t-a)$	e^{-as}	$(s > 0)$

For Periodic Functions:

Let $f(t)$ be a piecewise continuous periodic function defined on $0 \le t < \infty$, where $f(t)$ has period T. Then

$$\mathcal{L}\{f(t)\} = \frac{\int_0^T e^{-st}f(t)\,dt}{1-e^{-sT}}, \qquad s > 0$$

Convolution

$$(f * g)(t) = \int_0^t f(t-\lambda)g(\lambda)\,d\lambda$$

$$(f * g)(t) = \mathcal{L}^{-1}\{F(s)G(s)\}$$