Section6.4_Lay6e

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6.4 The Gram-Schmidt Process

The Gram-Schmidt Process is a technique by which, if you are given any basis for a subspace V, you can calculate an orthogonal basis for that subspace. The key step in the Gram-Schmidt Process is the calculation of the orthogonal projection of a vector $\mathbf{v}$ onto a subspace W, sometimes written as $\hat{\mathbf{v}} = \text{proj}_W \mathbf{v}$:

Orthogonal Projection

Let $\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_p\}$ be an orthogonal set of vectors in $\mathbb{R}^n$ and W be the subspace spanned by these vectors. Let $\mathbf{v}$ be any vector in $\mathbb{R}^n$.
The **orthogonal projection** of $\mathbf{v}$ onto W is given by

$$\text{proj}_W v = \hat{\mathbf{v}} = \underbrace{\frac{\mathbf{v}\cdot\mathbf{u}_1}{\mathbf{u}_1\cdot\mathbf{u}_1}\mathbf{u}_1}_{\text{proj}_{u_1} v} + \underbrace{\frac{\mathbf{v}\cdot\mathbf{u}_2}{\mathbf{u}_2\cdot\mathbf{u}_2}\mathbf{u}_2}_{\text{proj}_{u_2} v} + \cdots + \underbrace{\frac{\mathbf{v}\cdot\mathbf{u}_p}{\mathbf{u}_p\cdot\mathbf{u}_p}\mathbf{u}_p}_{\text{proj}_{u_p} v}$$

Example 6.4.1. Let $\mathbf{u}_1 = \begin{bmatrix} 2 \\ 5 \\ -1 \end{bmatrix}$, $\mathbf{u}_2 = \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}$, and $\mathbf{v} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ and $W = \text{Span}\{\mathbf{u}_1, \mathbf{u}_2\}$.

Find $\hat{\mathbf{v}} = \text{proj}_W \mathbf{v}$:

number

$$\hat{v} = \frac{v\cdot u_1}{u_1\cdot u_1}\vec{u_1} + \frac{v\cdot u_2}{u_2\cdot u_2}\vec{u_2}$$

$$v\cdot u_1 = 2 + 10 + (-3) = 9$$

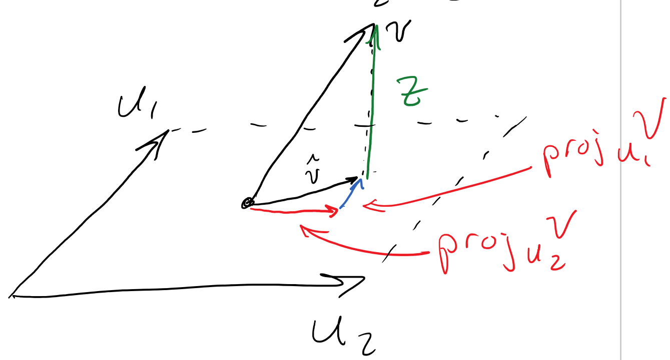
$$u_1\cdot u_1 = 30$$

$$v\cdot u_2 = 3$$

$$u_2\cdot u_2 = 6$$

$$\hat{v} = \frac{9}{30}\begin{bmatrix} 2 \\ 5 \\ -1 \end{bmatrix} + \frac{3}{6}\begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} -2/5 \\ 2 \\ 1/5 \end{bmatrix}$$



$$v = \hat{v} + z$$

The Gram-Schmidt Process

Let $\mathfrak{B} = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p\}$ be any linearly independent set of vectors and let V be the subspace spanned by $\mathfrak{B}$. We'll apply the Gram-Schmidt Process to find an orthogonal (or orthonormal) set of vectors which spans V.

1. We leave the first vector completely unchanged for now. That is, $\mathbf{w}_1 = \mathbf{v}_1$.
2. To find the other vectors, we calculate the projection of $\mathbf{v}_j$ onto the subspace spanned by $\{\mathbf{w}_1, \ldots, \mathbf{w}_{j-1}\}$,

the next vector in the original basis

$$\hat{\mathbf{v}}_j = \frac{\mathbf{v}_j \cdot \mathbf{w}_1}{\mathbf{w}_1 \cdot \mathbf{w}_1}\mathbf{w}_1 + \frac{\mathbf{v}_j \cdot \mathbf{w}_2}{\mathbf{w}_2 \cdot \mathbf{w}_2}\mathbf{w}_2 + \cdots + \frac{\mathbf{v}_j \cdot \mathbf{w}_{j-1}}{\mathbf{w}_{j-1} \cdot \mathbf{w}_{j-1}}\mathbf{w}_{j-1} \leftarrow \text{example } 6.4.1$$

then set $\mathbf{w}_j = \mathbf{v}_j - \hat{\mathbf{v}}_j$. (Optional: You may multiply $\mathbf{w}_j$ by the lowest common denominator of its components if that helps.)

$$v_j - \hat{v}_j = w_j$$

$\hat{v}_j$: the portion of v_j in $\{w_1, \ldots, w_{j-1}\}$

3. The set $\{\mathbf{w}_1, \ldots, \mathbf{w}_p\}$ is an orthogonal basis for W.

If you want an orthonormal basis for W then continue as follows: $\rightarrow$ orthogonal and length 1.

4. Once the vectors $\{\mathbf{w}_1, \ldots, \mathbf{w}_p\}$ have been computed, scale them to a length of 1: $\mathbf{u}_j = \dfrac{\mathbf{w}_j}{||\mathbf{w}_j||}$
5. The set $\{\mathbf{u}_1, \ldots, \mathbf{u}_p\}$ is an orthonormal basis for V

Example 6.4.2. Find an orthonormal basis for for $W = \text{Span}\left\{\underbrace{\begin{bmatrix}0\\4\\2\end{bmatrix}}_{v_1}, \underbrace{\begin{bmatrix}5\\6\\-7\end{bmatrix}}_{v_2}\right\}$

$$v_1 = \begin{bmatrix}0\\4\\2\end{bmatrix} \quad v_2 = \begin{bmatrix}5\\6\\-7\end{bmatrix}$$

$$w_1 = \begin{bmatrix}0\\4\\2\end{bmatrix}$$

$$w_2 = v_2 - \hat{v}_2 = v_2 - \frac{v_2 \cdot w_1}{w_1 \cdot w_1}\vec{w_1}$$

$$v_2 \cdot w_1 = 0 + 24 - 14 = 10$$

$$w_1 \cdot w_1 = 0 + 16 + 4 = 20$$

$$= \begin{bmatrix}5\\6\\-7\end{bmatrix} - \frac{1}{2}\begin{bmatrix}0\\4\\2\end{bmatrix} = \begin{bmatrix}5\\4\\-8\end{bmatrix}$$

$||w_1|| = \sqrt{20}$

$||w_2|| =$

$$\boxed{\text{orthogonal basis is } W = \{w_1, w_2\} = \left\{\begin{bmatrix}0\\4\\2\end{bmatrix}, \begin{bmatrix}5\\4\\-8\end{bmatrix}\right\}}$$

$$||w_1|| = \sqrt{20} \qquad ||w_2|| = \sqrt{25 + 16 + 64} = \sqrt{105}$$

$$\boxed{\text{orthonormal basis} = \left\{\begin{bmatrix}0\\4/\sqrt{20}\\2/\sqrt{20}\end{bmatrix}, \begin{bmatrix}5/\sqrt{105}\\4/\sqrt{105}\\-8/\sqrt{105}\end{bmatrix}\right\}}$$

Example 6.4.3. Find an orthogonal basis for Col A where

$$A = \begin{bmatrix} 3 & -5 & 1 \\ 1 & 1 & 1 \\ -1 & 5 & -2 \\ 3 & -7 & 8 \end{bmatrix} \qquad ans = \left\{ \begin{bmatrix} 3 \\ 1 \\ -1 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \\ 3 \\ -1 \end{bmatrix}, \begin{bmatrix} -3 \\ 1 \\ 1 \\ 3 \end{bmatrix} \right\}$$

$$v_1 = \begin{bmatrix} 3 \\ 1 \\ -1 \\ 3 \end{bmatrix}, \quad v_2 = \begin{bmatrix} -5 \\ 1 \\ 5 \\ -7 \end{bmatrix}, \quad v_3 = \begin{bmatrix} 1 \\ 1 \\ -2 \\ 8 \end{bmatrix}$$

$$w_1 = v_1 = \begin{bmatrix} 3 \\ 1 \\ -1 \\ 3 \end{bmatrix}$$

$$v_2 \cdot w_1 = -15 + 1 - 5 - 21 = -40$$

$$w_1 \cdot w_1 = 9 + 1 + 1 + 9 = 20$$

$$w_2 = v_2 - \text{proj}_{w_1} v_2 = v_2 - \frac{v_2 \cdot w_1}{w_1 \cdot w_1} \vec{w_1}$$

$$= \begin{bmatrix} -5 \\ 1 \\ 5 \\ -7 \end{bmatrix} - (-2) \begin{bmatrix} 3 \\ 1 \\ -1 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 3 \\ -1 \end{bmatrix} = w_2$$

$$\frac{v_3 \cdot w_1}{w_1 \cdot w_1} = \frac{3}{2}$$

$$\frac{v_3 \cdot w_2}{w_2 \cdot w_2} = -\frac{1}{2}$$

$$w_3 = v_3 - \text{proj}_{w_1} v_3 - \text{proj}_{w_2} v_3$$

$$= v_3 - \frac{v_3 \cdot w_1}{w_1 \cdot w_1} \vec{w_1} - \frac{v_3 \cdot w_2}{w_2 \cdot w_2} \vec{w_2}$$

$$= \begin{bmatrix} 1 \\ 1 \\ -2 \\ 8 \end{bmatrix} - \frac{3}{2} \begin{bmatrix} 3 \\ 1 \\ -1 \\ 3 \end{bmatrix} - \left(-\frac{1}{2}\right) \begin{bmatrix} 1 \\ 3 \\ 3 \\ -1 \end{bmatrix} = \begin{bmatrix} -3 \\ 1 \\ 1 \\ 3 \end{bmatrix} = w_3$$

$w_2 \cdot w_2 = 6 + 6 - 6 - 6 = 0$ ✓

Example 6.4.4. Find an orthogonal basis for

$$\mathfrak{B} = \text{Span}\left\{ \underset{v_1}{\begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \\ 1 \end{bmatrix}}, \underset{v_2}{\begin{bmatrix} 2 \\ 1 \\ 4 \\ -4 \\ 2 \end{bmatrix}}, \underset{v_3}{\begin{bmatrix} 5 \\ -4 \\ -3 \\ 7 \\ 1 \end{bmatrix}} \right\} \qquad \boxed{ans = \left\{ \begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \\ 3 \\ -3 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 2 \\ 2 \\ -2 \end{bmatrix} \right\}}$$

$$W_1 = v_1 = \begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \\ 1 \end{bmatrix}$$

$$v_2 \cdot w_1 = 2 - 1 - 4 - 4 + 2 = -5$$

$$w_1 \cdot w_1 = 5$$

$$W_2 = v_2 - \text{proj}_{w_1} v_2 = v_2 - \frac{v_2 \cdot w_1}{w_1 \cdot w_1} \vec{w_1}$$

$$= \begin{bmatrix} 2 \\ 1 \\ 4 \\ -4 \\ 2 \end{bmatrix} - (-1) \begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \\ 3 \\ -3 \\ 3 \end{bmatrix} = W_2$$

$$w_1 \cdot w_1 = 5$$

$$v_3 \cdot w_1 = 5 + 4 + 3 + 7 + 1 = 20$$

$$v_3 \cdot w_2 = 15 + 0 - 9 - 21 + 3 = -12$$

$$w_2 \cdot w_2 = 9 + 0 + 9 + 9 + 9 = 36$$

$$W_3 = v_3 - \text{proj}_{w_1} v_3 - \text{proj}_{w_2} v_3$$

$$= v_3 - \frac{v_3 \cdot w_1}{w_1 \cdot w_1} \vec{w_1} - \frac{v_3 \cdot w_2}{w_2 \cdot w_2} \vec{w_2}$$

$$= \begin{bmatrix} 5 \\ -4 \\ -3 \\ 7 \\ 1 \end{bmatrix} - 4 \begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \\ 1 \end{bmatrix} + \frac{1}{3} \begin{bmatrix} 3 \\ 0 \\ 3 \\ -3 \\ 3 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 2 \\ 2 \\ -2 \end{bmatrix} = W_3$$

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