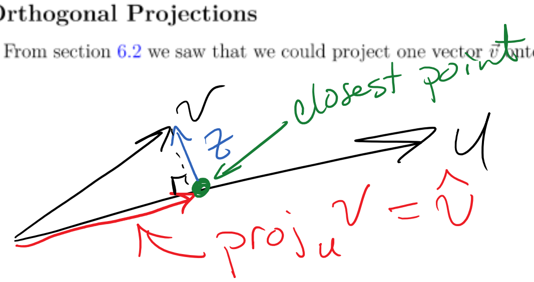
# Section6.3_Lay6e

Thursday, November 10, 2022 11:02 AM



## 6.3 Orthogonal Projections

**Recall:** From section 6.2 we saw that we could project one vector $\vec{v}$ onto another vector $\vec{u}$: $\text{proj}_{\vec{u}}\vec{v}$.




$\hat{v} + z = v$

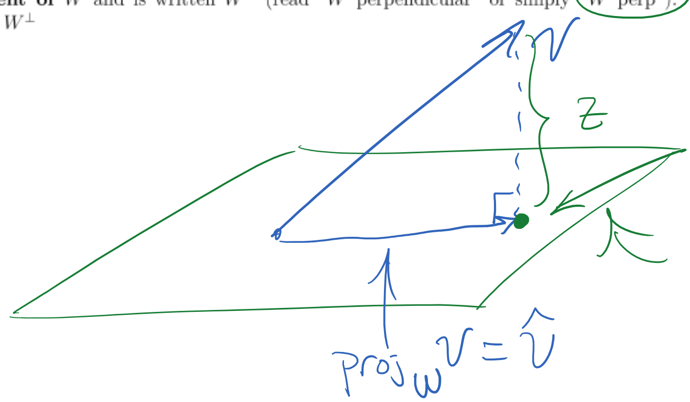
$z = v - \hat{v}$

Now we would like to project vector $\vec{v}$ onto a subspace $W$ of $\mathbb{R}^n$: $\hat{v} = \text{proj}_W \vec{v}$. $\vec{v}$ can be written as a component in $W$ ($\hat{v}$) and a component perpendicular to $W$ ($z$).

$\langle a, b, c \rangle = \vec{v}$

$(a, b, c)$ closest point

**Definition 6.8.** The space perpendicular to $W = \text{span}\{u_1, u_2, \ldots, u_p\}$ is called ~~the~~ **orthogonal complement of** $W$ and is written $W^\perp$ (read "$W$ perpendicular" or simply "$W$ perp"). Note: $\mathbb{R}^n = W + W^\perp$




closest point in W to $\vec{v}$

W is a subspace of $\mathbb{R}^3$

$v = \hat{v} + z$

in W

$W^\perp$

**Example 6.3.1.** Show that $W^\perp$ is a subspace of $\mathbb{R}^n$.

Start with $u \in W$ and show

1. zero vector (is $0 \in W^\perp$?)
2. closed under addition. (for $v \in W^\perp$ and $x \in W^\perp$ is $v + x \in W^\perp$?)
3. all other properties inherited from $\mathbb{R}^n$

#1 $u \cdot 0 = 0$ yes $\Big\}$ 0 is in $W^\perp$



#2 $v \in W^\perp$, $x \in W^\perp$ IS $x + v \in W^\perp$? Need $(x+v)\cdot u = 0$

$x \cdot u + v \cdot u$

$0 + 0 = 0$ ✓



**Orthogonal Basis**

**Theorem 6.5.** Suppose $\{u_1, u_2, \ldots, u_n\}$ is an orthogonal basis for $\mathbb{R}^n$ and $W = \{u_1, u_2, \ldots, u_p\}$ then any vector $y \in \mathbb{R}^n$

$$y = c_1u_1 + c_2u_2 + \cdots + c_pu_p + \boxed{c_{p+1}u_{p+1} + \cdots + c_nu_n}$$

$W = \{u_1, u_2, \ldots, u_p\}$ then any vector $y \in \mathbb{R}^n$

$$y = c_1u_1 + c_2u_2 + \cdots + c_pu_p + \boxed{c_{p+1}u_{p+1} + \cdots + c_nu_n}$$

In $W$ $\qquad$ $W^{\perp}$

are given by

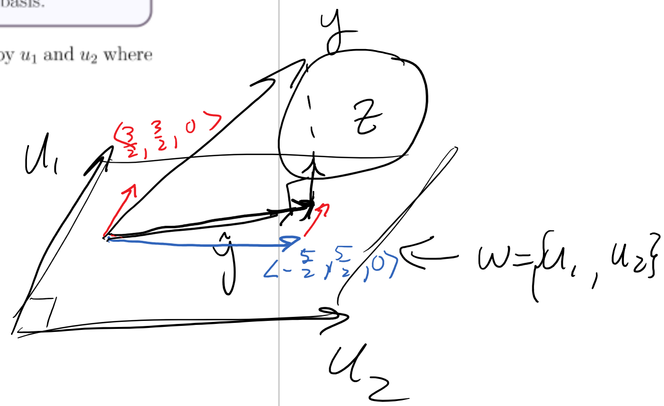
$$c_i = \frac{u_i \cdot y}{u_i \cdot u_i}$$

**Note:** The weights $c_i$ are the projections onto each vector in the orthogonal basis.

**Example 6.3.2.** Find the orthogonal projection of $y$ onto the subspace spanned by $u_1$ and $u_2$ where

$$y = \begin{bmatrix} -1 \\ 4 \\ 3 \end{bmatrix}, \qquad u_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \qquad u_2 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$$

Find $\hat{y} = \text{proj}_W y = c_1u_1 + c_2u_2$




$$c_1 = \frac{u_1 \cdot y}{u_1 \cdot u_1} = \frac{3}{2} \qquad c_2 = \frac{u_2 \cdot y}{u_2 \cdot u_2} = \frac{5}{2}$$

$$u_1 \cdot y = \langle 1, 1, 0\rangle \cdot \langle -1, 4, 3\rangle = -1 + 4 + 0 = 3$$

$$u_1 \cdot u_1 = \langle 1, 1, 0\rangle \cdot \langle 1, 1, 0\rangle = 1 + 1 = 2$$

$$u_2 \cdot y = \langle -1, 1, 0\rangle \cdot \langle -1, 4, 3\rangle = 1 + 4 = 5$$

$$u_2 \cdot u_2 = \langle -1, 1, 0\rangle \cdot \langle -1, 1, 0\rangle = 1 + 1 = 2$$

$$\hat{y} = c_1u_1 + c_2u_2 = \frac{3}{2}\langle 1, 1, 0\rangle + \frac{5}{2}\langle -1, 1, 0\rangle$$

$\hat{y} = \langle -1, 4, 0\rangle$ Also closest point $\langle -1, 4, 0\rangle$

$$z = y - \hat{y} = \langle -1, 4, 3\rangle - \langle -1, 4, 0\rangle = \langle 0, 0, 3\rangle$$

$\|z\| = 3$ shortest distance

**Orthogonal Basis**

**Theorem 6.5.** Suppose $\{u_1, u_2, \ldots, u_n\}$ is an orthogonal basis for $\mathbb{R}^n$ and $W = \{u_1, u_2, \ldots, u_p\}$ then any vector $y \in \mathbb{R}^n$

$$y = c_1u_1 + c_2u_2 + \cdots + c_pu_p + \boxed{c_{p+1}u_{p+1} + \cdots + c_nu_n}$$

In $W$ $\qquad$ $W^{\perp}$

are given by

$$c_i = \frac{u_i \cdot y}{u_i \cdot u_i}$$

**Note:** The weights $c_i$ are the projections onto each vector in the orthogonal basis.

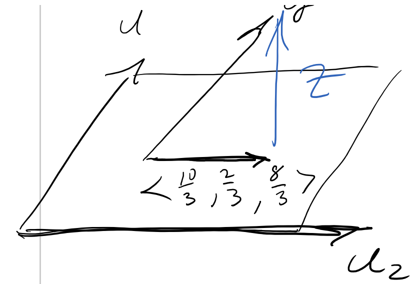


**Example 6.3.3.** $W$ is the subspace spanned by

$$u_1 = \begin{bmatrix} 1 \\ 3 \\ -2 \end{bmatrix}, \qquad u_2 = \begin{bmatrix} 5 \\ 1 \\ 4 \end{bmatrix}.$$

$$u_1 = \begin{bmatrix} 1 \\ 3 \\ -2 \end{bmatrix}, \qquad u_2 = \begin{bmatrix} 5 \\ 1 \\ 4 \end{bmatrix}.$$

Write $y = \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix}$ as a sum of a vector in $W$ and a vector in $W^\perp$ (ie. $y = \hat{y} + z$)




$u_1 \cdot y = 0$

$u_1 \cdot u_1 = 14$

$u_2 \cdot y = 28$

$u_2 \cdot u_2 = 42$

$$\hat{y} = \frac{u_1 \cdot y}{u_1 \cdot u_1} u_1 + \frac{u_2 \cdot y}{u_2 \cdot u_2} u_2 = 0 \langle 1, 3, -2 \rangle + \frac{2}{3} \langle 5, 1, 4 \rangle$$

$$\boxed{\hat{y} = \begin{bmatrix} \frac{10}{3} \\ \frac{2}{3} \\ \frac{8}{3} \end{bmatrix}}$$

$$z = y - \hat{y} = \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix} - \begin{bmatrix} 10/3 \\ 2/3 \\ 8/3 \end{bmatrix}$$

$$\boxed{z = \begin{bmatrix} -7/3 \\ 7/3 \\ 7/3 \end{bmatrix}}$$

$$\boxed{y = \begin{bmatrix} 10/3 \\ 2/3 \\ 8/3 \end{bmatrix} + \begin{bmatrix} -7/3 \\ 7/3 \\ 7/3 \end{bmatrix}}$$

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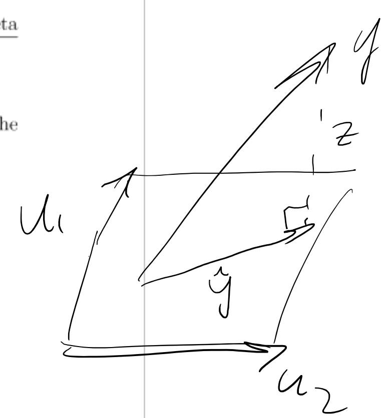
**Example 6.3.4.** Find the closest point ($\hat{y}$) and the shortest distance ($||z||$) to $\vec{y} = \begin{bmatrix} 3 \\ 1 \\ 5 \\ 1 \end{bmatrix}$ in the subspace

$$W = \{u_1, u_2\} = \left\{ \begin{bmatrix} 3 \\ 1 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix} \right\}.$$




$u_1 \cdot y = 6$

$u_1 \cdot u_1 = 12$

$u_2 \cdot y = 6$

$u_2 \cdot u_2 = 4$

closest point:

$$\hat{y} = \frac{u_1 \cdot y}{\ } u_1 + \frac{u_2 \cdot y}{\ } u_2$$

closest point:

$$\hat{y} = \frac{u_1 \cdot y}{u_1 \cdot u_1} u_1 + \frac{u_2 \cdot y}{u_2 \cdot u_2} u_2$$

$$= \frac{1}{2}\langle 3, 1, -1, 1\rangle + \frac{3}{2}\langle 1, -1, 1, -1\rangle$$

$$= \langle 3, -1, 1, -1\rangle$$ closest point

$$z = y - \hat{y} = \begin{bmatrix} 3 \\ 1 \\ 5 \\ 1 \end{bmatrix} - \begin{bmatrix} 3 \\ -1 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \\ 4 \\ 2 \end{bmatrix}$$

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$$\|z\| = \sqrt{0 + 2^2 + 4^2 + 2^2}$$

$$\|z\| = \sqrt{24}$$