Section6.2_Lay6e

Thursday, November 10, 2022 11:02 AM

6.2 Orthogonal Sets

Orthogonal Sets

Definition 6.4. A set of vectors $\{v_1, v_2, \ldots, v_n\}$ is an **orthogonal set** if each pair of distinct vectors is orthogonal. ie:

$$v_i \cdot v_j = 0 \qquad \text{for each } i \neq j$$

Definition 6.5. An **orthogonal basis** is a basis that is also an orthogonal set.

Definition 6.6. An **orthonormal basis (set)** is an orthogonal basis (set) of unit vectors (length 1).

Perpendicular ← Length 1

Theorem 6.2. Orthogonal sets are linearly independent.

Ex: $\hat{i} = \langle 1, 0, 0 \rangle$

$\hat{j} = \langle 0, 1, 0 \rangle$

$\hat{k} = \langle 0, 0, 1 \rangle$

Example 6.2.1. Show that the following vectors form an orthogonal set.

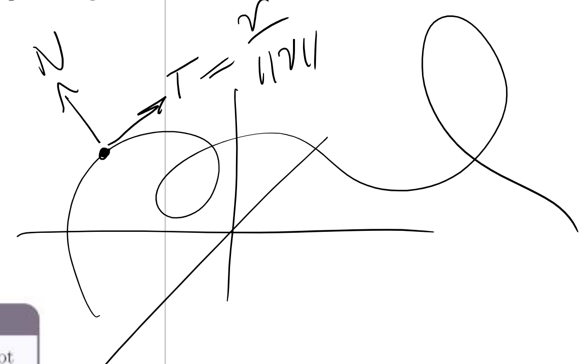
$$u_1 =< 1, -2, 1 >, \quad u_2 =< 0, 1, 2 >, \quad u_3 =< -5, -2, 1 >$$

$u_1 \cdot u_2 = 0 - 2 + 2 = 0$ $\quad u_1 \cdot u_3 = -5 + 4 + 1 = 0$ $\quad u_2 \cdot u_3 = -2 + 2 = 0$ ✓

Example 6.2.2. Construct an orthonormal basis from

$$\frac{u_1 =< 1, -2, 1 >}{\sqrt{6}}, \quad \frac{u_2 =< 0, 1, 2 >}{\sqrt{5}}, \quad \frac{u_3 =< -5, -2, 1 >}{\sqrt{30}}$$

$\|u_1\| = \sqrt{6}$



6.2.1 Orthogonal Projection of One Vector onto Another

Projection of One Vector onto Another.

Theorem 6.3. The angle θ between two vectors, u and v, can be calculated using the dot product:

$$u \cdot v = \|u\|\|v\| \cos\theta$$

$$\cos\theta = \frac{u \cdot v}{\|u\|\|v\|}$$

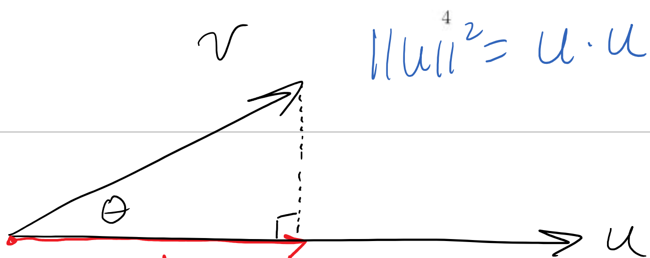
Definition 6.7. We can use the angle and the dot product to find the **projection of** v **onto** u.

$$\hat{v} = \text{proj}_u v = \left(\frac{u \cdot v}{\|u\|^2}\right) u = \left(\frac{u \cdot v}{u \cdot u}\right) u$$

vector

number

$\|u\|^2 = u \cdot u$



$\hat{v} = \text{proj}_u v$

length $\hat{v} = \|v\| \boxed{\cos\theta}$

direction $\hat{v}$ = direction $u = \frac{u}{\|u\|}$

unit vector = direction

magnitude

$\hat{v} = \|v\| \cos\theta \left(\frac{\vec{u}}{\|u\|}\right)$ ← only vector

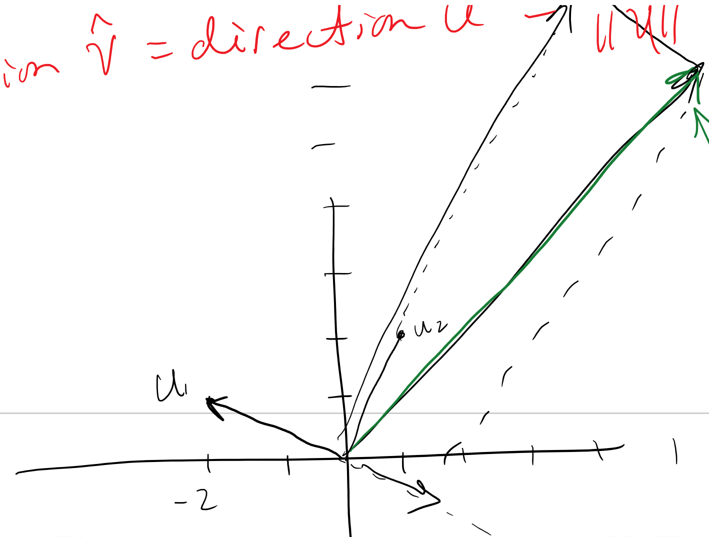
$= \cancel{\|v\|} \frac{u \cdot v}{\|u\| \cancel{\|v\|}} \left(\frac{\vec{u}}{\|u\|}\right)$

a number

$= \frac{u \cdot v}{\|u\|^2} (\vec{u})$

number vector

direction $\hat{v}$ = direction $u \to \|u\|$ number, vector



Example 6.2.3. Find the projection of $v = \begin{bmatrix} 5 \\ 6 \end{bmatrix}$ onto

1. $u_1 = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$ and $\|u_1\|^2 = 5$
2. $u_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ $\|u_2\|^2 = 5$
3. What is the sum of the two projections?

$u_1 \cdot v = \langle -2, 1 \rangle \cdot \langle 5, 6 \rangle = -4$

$u_2 \cdot v = \langle 1, 2 \rangle \cdot \langle 5, 6 \rangle = 17$

$$\text{Proj}_{u_1} v = \frac{u_1 \cdot v}{\|u_1\|^2} \cdot \vec{u}_1 = \frac{-4}{5} \langle -2, 1 \rangle = \left\langle \frac{8}{5}, \frac{-4}{5} \right\rangle$$

$$\text{Proj}_{u_2} v = \frac{u_2 \cdot v}{\|u_2\|^2} \vec{u}_2 = \frac{17}{5} \langle 1, 2 \rangle = \left\langle \frac{17}{5}, \frac{34}{5} \right\rangle$$

$$\text{Proj}_{u_1} v + \text{Proj}_{u_2} v = \begin{bmatrix} 5 \\ 6 \end{bmatrix}$$

Orthogonal Basis

Theorem 6.4. Let $\{u_1, u_2, \ldots, u_p\}$ be an orthogonal basis for a subspace W of $\mathbb{R}^n$. For each $y \in W$, the weights in

$$y = c_1 u_1 + c_2 u_2 + \cdots + c_p u_p$$

are given by

New basis → $$c_i = \frac{u_i \cdot y}{u_i \cdot u_i}$$

Note: The weights c_i are the projections onto each vector in the orthogonal basis.

$y = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ \vdots \\ y_p \end{bmatrix}$ ← $y \in \mathbb{R}^n$ standard bases

components of the projections.

Example 6.2.4. Write $y = \begin{bmatrix} 6 \\ 1 \\ -8 \end{bmatrix}$ as a linear combination of

$$u_1 = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}, \quad u_2 = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}, \quad u_3 = \begin{bmatrix} -5 \\ -2 \\ 1 \end{bmatrix}$$

We showed in example 6.2.1 that this is an orthogonal basis.

From thm 6.4

$$y = c_1 u_1 + c_2 u_2 + c_3 u_3$$

$$\begin{bmatrix} 6 \\ 1 \\ -8 \end{bmatrix} = \frac{y \cdot u_1}{u_1 \cdot u_1} (\vec{u}_1) + \frac{y \cdot u_2}{u_2 \cdot u_2} (\vec{u}_2) + \frac{y \cdot u_3}{u_3 \cdot u_3} (\vec{u}_3)$$

$u_1 \cdot u_1 = \langle 1, -2, 1 \rangle \cdot \langle 1, -2, 1 \rangle = 6$

$$y \cdot u_1 = \langle 6, 1, -8 \rangle \cdot \langle 1, -2, 1 \rangle = 6 - 2 - 8 = -4$$

$$y \cdot u_2 = \langle 6, 1, -8 \rangle \cdot \langle 0, 1, 2 \rangle = 0 + 1 - 16 = -15$$

$$y \cdot u_3 = \langle 6, 1, -8 \rangle \cdot \langle -5, -2, 1 \rangle = -30 - 2 - 8 = -40$$

$$u_1 \cdot u_1 = \langle 1, -2, 1 \rangle \cdot \langle 1, -2, 1 \rangle = 6$$

$$u_2 \cdot u_2 = \langle 0, 1, 2 \rangle \cdot \langle 0, 1, 2 \rangle = 5$$

$$u_3 \cdot u_3 = \langle -5, -2, 1 \rangle \cdot \langle -5, -2, 1 \rangle = 30$$

$$\boxed{y = \frac{-4}{6} u_1 + \frac{-15}{5} u_2 + \frac{-40}{30} u_3}$$