Section6.1_Lay6e

Thursday, November 10, 2022 11:01 AM

6.1 Inner Product, Length, and Orthogonality

6.1.1 Multiplying Vectors

There are two ways to multiply vectors u and v:

1. the **cross product** $u \times v$. We will not be discussing cross products in this class.
2. the **dot product** $u \cdot v$. The dot product is also called the **inner product**. number

Inner Product

Definition 6.1. The **inner product** (or **dot product**) of two $x \times 1$ vectors u and v

$$u = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix} \qquad v = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$$

is the product $u \cdot v = u^{\mathrm{T}} v$.

$$u \cdot v = \begin{bmatrix} u_1 & u_2 & \cdots & u_n \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} = u_1 v_1 + u_2 v_2 + \cdots + u_n v_n$$

IMPORTANT: Notice that when you take the dot product of two vectors you have a scalar answer. The dot product is a single number.

Example 6.1.1. Find $u \cdot v$ for $u = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ and $v = \begin{bmatrix} -1 \\ 2 \\ -7 \end{bmatrix}$

$$u \cdot v = 1(-1) + 2(2) + 3(-7)$$
$$= -1 + 4 - 21 = \boxed{-18}$$

Properties of the Inner Product

Properties of the Inner Product: Let u, v, and w be vectors in $\mathbb{R}^n$, and let c be a scalar. Then

1. $u \cdot v = v \cdot u$
2. $(u + v) \cdot w = u \cdot w + v \cdot w$
3. $(cu) \cdot v = c(u \cdot v) = u \cdot (cv)$
4. $u \cdot u \geq 0$, and $u \cdot u = 0 \Leftrightarrow u = 0$

$$u \cdot u = u_1^2 + u_2^2 + \cdots + u_n^2$$

Properties 2 and 3 can be combined into

$$(c_1 u_1 + \cdots + c_p u_p) \cdot w = c_1(u_1 \cdot w) + \cdots + c_p(u_p \cdot w)$$

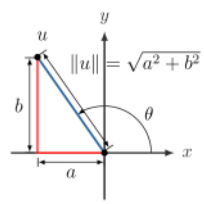
6.1.2 Length of a Vector

Length of a Vector

Definition 6.2. The **length** (or **norm**) of a vector v is the nonnegative scalar $\|v\|$ defined by

$$\|v\| = \sqrt{v \cdot v} = \sqrt{v_1^2 + v_2^2 + \cdots + v_n^2}, \qquad \text{and } \|v\|^2 = v \cdot v$$

A **unit vector** is a vector of length 1.
If v is a vector the unit vector in the direction of v is $\frac{v}{\|v\|}$. ← length = 1
The process of changing a vector v into a unit vector is called **normalizing** v.



$\|v\|^2 = v \cdot v$

magnitude

Example 6.1.2. For $v = < -1, 2, -7 >$

1. Find the length of v. $\|v\| = \sqrt{(1)^2 + (2)^2 + 7^2} = \sqrt{54}$
2. Find a unit vector in the direction of v.
3. Write v as (magnitude) · (direction) where the direction is a unit vector.

#2) $\frac{v}{\|v\|} = \left\langle -\frac{1}{\sqrt{54}}, \frac{2}{\sqrt{54}}, \frac{-7}{\sqrt{54}} \right\rangle$

#3) $v = \sqrt{54} \underbrace{\left\langle -\frac{1}{\sqrt{54}}, \frac{2}{\sqrt{54}}, \frac{-7}{\sqrt{54}} \right\rangle}_{\text{unit vector = direction.}}$

Example 6.1.3. Find the dot product between $u = < 12, 3, -5 >$ and $v = < 2, -3, 3 >$

$u \cdot v = 12(2) + 3(-3) + (-5)(3) = 0$

Orthogonal Vectors

Definition 6.3. Two vectors u and v are **orthogonal (perpendicular)** to each other if $u \cdot v = 0$.

Theorem 6.1. Two vectors u and v are orthogonal if and only if $\|u + v\|^2 = \|u\|^2 + \|v\|^2$. (ie. the Pythagorean theorem is true)