Section5.5_Lay6e

Monday, October 31, 2022 2:40 PM

5.5 Complex Eigenvalues

5.5.1 Complex Arithmetic

Q: What is a complex number?

A: It is a number of the form $a + bi$ where a and b are real numbers and $i^2 = -1$.

- a is called the **real part**
- b is called the **imaginary part**.
- The **conjugate** of $a + bi$ is $a - bi$.

$a + bi$

$a - bi$ complex conjugate.

Properties

1. $a + b\,i = c + d\,i \quad \Leftrightarrow \quad a = c$ and $b = d$
2. $(a + b\,i) + (c + d\,i) = (a + c) + (b + d)\,i$
3. $(a + b\,i)\cdot(c + d\,i) = ac + ad\,i + bc\,i + bd\,i^2 = (ac - bd) + (ad + bc)\,i$
4. $(a + b\,i)(a - b\,i) = a^2 + b^2$

Example 5.5.1.

a. $\frac{i}{(3+i)}\frac{(3-i)}{(3-i)} = \boxed{\frac{3i+1}{10}} = \boxed{\frac{1}{10} + \frac{3}{10}i}$

b. $\frac{(3-5i)(2+i)}{(2-i)(2+i)} = \frac{6+3i-10i-5i^2}{5} = \frac{11-7i}{5} = \boxed{\frac{11}{5} - \frac{7}{5}i}$

c. ~~$(3 - \sqrt{-4}) + (-8 + \sqrt{-25})$~~

~~$(\sqrt{-5})(\sqrt{-5}) = \sqrt{(-5)(-5)} = \sqrt{25} = 5$~~

$(\sqrt{5}\,i)(\sqrt{5}\,i) = (\sqrt{5})^2\, i^2 = -5$

d. $\frac{1}{3i}\cdot\frac{i}{i} = \frac{i}{-3}$

ALWAYS MAKE COMPLEX NUMBER FIRST

$\sqrt{a}\,\sqrt{b} = \sqrt{ab}$

5.5.2 Complex Numbers in Equations

Example 5.5.2. Solve for x: $x^2 + 1 = 0$

$x^2 = -1$

$\boxed{x = \pm i}$

$x^2 = -1$

$x = \pm\sqrt{-1}$

$x = \pm i$

Example 5.5.3. Solve the system of equations

$$\begin{aligned} (-1+i)\quad (1+i)x_1 &+ (-1+i)\, i x_2 = 0 \quad (-1+i) \\ + \quad (1-i)x_1 &+ (1+i)x_2 = 0 \end{aligned}$$

$-ix_2 + i^2x_2 + (1+i)x_2 = 0$

$-ix_2 - x_2 + x_2 + ix_2 = 0$

$0 = 0$

$x_1 = -i x_2$

$x_2 = x_2$

$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -i \\ 1 \end{bmatrix} x_2$

Example 5.5.4. Find the eigenvalues and eigenvectors of $\begin{bmatrix} 5 & -2 \\ 1 & 3 \end{bmatrix}$

$\det(A - \lambda I) = 0$

$$\begin{vmatrix} 5-\lambda & -2 \\ 1 & 3-\lambda \end{vmatrix} = (5-\lambda)(3-\lambda) + 2$$

$= 15 - 5\lambda - 3\lambda + \lambda^2 + 2$

$= \lambda^2 - 8\lambda + 17 = 0$

$$\lambda = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{8 \pm \sqrt{64 - 4(1)(17)}}{2(1)} = \frac{8 \pm \sqrt{-4}}{2} = \frac{8 \pm 2i}{2} = 4 \pm i$$

$\lambda = 4 + i$ solve $(A - \lambda I)x = 0$ for x

$$\left[\begin{array}{cc|c} 5-(4+i) & -2 & 0 \\ 1 & 3-(4+i) & 0 \end{array}\right] \longrightarrow \left[\begin{array}{cc|c} 1-i & -2 & 0 \\ 1 & -1-i & 0 \end{array}\right] \xrightarrow{RR} \left[\begin{array}{cc|c} 1 & -1-i & 0 \\ 0 & 0 & 0 \end{array}\right]$$

$R_2(1-i)$: $1-i$ $\quad (-1-i)(1-i) = -1 + i^2 = -2$

$1x_1 + (-1-i)x_2 = 0$

$x_1 = (1+i)x_2$

$x_2 = x_2$

$$\boxed{X = \begin{bmatrix} 1+i \\ 1 \end{bmatrix} \text{ for } \lambda = 4+i}$$

$$(1-i)x_1 - 2x_2 = 0 \qquad x_1 = \frac{2}{1-i}\frac{(1+i)}{(1+i)}x_2 = \frac{2(1+i)}{2}x_2$$

$$(1-i)x_1 = 2x_2 \qquad x_2 = x_2$$

$$x^2 + 25 = 0$$

Example 5.5.5. Find the eigenvalues and eigenvectors of $\begin{bmatrix} -5 & -5 \\ 5 & -5 \end{bmatrix}$

$$\begin{vmatrix} -5-\lambda & -5 \\ 5 & -5-\lambda \end{vmatrix} = (-5-\lambda)^2 + 25 = 0$$

$$\sqrt{(-5-\lambda)^2} = \pm\sqrt{-25}$$

$$-5-\lambda = \pm 5i$$

$$\lambda = -5 \pm 5i$$

$\lambda = -5+5i$ solve $(A-\lambda I)x = 0$

$$\left[\begin{array}{cc|c} -5-(-5+5i) & -5 & 0 \\ 5 & -5-(-5+5i) & 0 \end{array}\right] = \left[\begin{array}{cc|c} -5i & -5 & 0 \\ 5 & -5i & 0 \end{array}\right]$$

$$\xrightarrow{RR} \begin{bmatrix} +i & +1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad \underline{i}x_1 + x_2 = 0$$

$$x_1 = x_1$$

$$x_2 = -ix_1$$

$$\boxed{x = \begin{bmatrix} 1 \\ -i \end{bmatrix}}$$

$$\boxed{E_{\lambda=-5+5i} = \left\{\begin{bmatrix} 1 \\ -i \end{bmatrix}\right\}}$$

5.5.3 Polar Form of a Complex Number

Euler's Formula

Euler's Formula

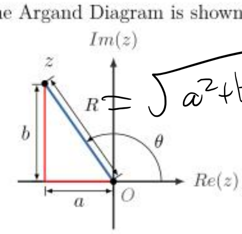
$$\boxed{e^{i\theta} = \cos\theta + i\sin\theta}$$
$$e^{-i\theta} = \cos(-\theta) + i\sin(-\theta)$$
$$= \cos\theta - i\sin\theta$$

$$e^{i\pi} = -1$$

Euler's formula allows us to plot complex numbers on the Re-Im plane. This is called an **Argand Diagram**. For example for a complex number of the form

$$z = a + bi = Re^{i\theta}$$

the Argand Diagram is shown below.



$$R = \sqrt{a^2+b^2}$$

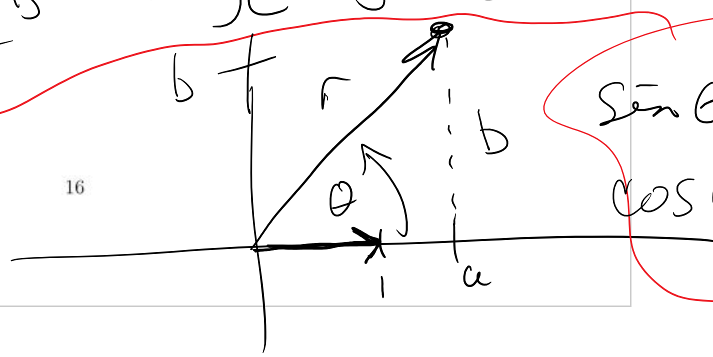
$$(a-\lambda)^2 + b^2 = 0$$

Example 5.5.6. If $C = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$, where a and b are nonzero real numbers, then the eigenvalues of C are $\lambda = a \pm bi$. Show that C can be written as a scale factor matrix $\begin{bmatrix} r & 0 \\ 0 & r \end{bmatrix}$ where $r = |\lambda|$ and a rotation matrix $\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$ where θ is the angle between the positive x-axis and the vector $[a, b]$.

$$\begin{bmatrix} a & -b \\ b & a \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} a \\ b \end{bmatrix}$$

Solution:

$$C = r\begin{bmatrix} a/r & -b/r \\ b/r & a/r \end{bmatrix}$$



$$\sin\theta = \frac{b}{r}$$
$$\cos\theta = \frac{a}{r}$$

$$r = |\lambda| = \sqrt{a^2+b^2}$$

$$\frac{\begin{bmatrix} -5 & -5 \\ 5 & -5 \end{bmatrix}}{r}$$

$$\lambda = -5 \pm 5i$$
$$|\lambda| = r = 5\sqrt{2}$$

$$\begin{bmatrix} \frac{-5}{5\sqrt{2}} & \frac{-5}{5\sqrt{2}} \\ \frac{5}{\ } & \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

$$\begin{bmatrix} 5\sqrt{2} & 5\sqrt{2} \\ \frac{5}{5\sqrt{2}} & -\frac{5}{5\sqrt{2}} \end{bmatrix} = \begin{bmatrix} & \\ \sin\theta & \cos\theta \end{bmatrix}$$

$$\cos\theta = -\frac{5}{5\sqrt{2}} \quad \frac{\text{adj}}{\text{hyp}} \qquad \sin\theta = \frac{5}{5\sqrt{2}}$$