Section5.3_Lay6e

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5.3 Diagonalization

5.3.1 Diagonal matrices

Recall:

The **characteristic equation** of a square matrix A, formally notated $\det(A - \lambda I) = 0$, is the equation obtained by subtracting the variable λ from the entries along the main diagonal of A, then taking the determinant and setting it equal to zero.

The roots of this polynomial equation are the **eigenvalues** of A. If a root is repeated k times, we say that eigenvalue has **algebraic multiplicity** k.

Example 5.3.1. Find the eigenvalues and eigenvectors of $A = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix}$

$$\det(A - \lambda I) = \boxed{\begin{vmatrix} 2-\lambda & 1 & 0 \\ 0 & 2-\lambda & 1 \\ 0 & 0 & 2-\lambda \end{vmatrix}} = (2-\lambda)^3 = 0 \qquad \lambda = 2, \ 3\text{times}$$

Eigenvectors: solve $(A - 2I)x = 0$ for x.

$$\left[\begin{array}{ccc|c} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \qquad \begin{aligned} x_1 &= x_1 \\ x_2 &= 0 \\ x_3 &= 0 \end{aligned} \qquad E_{\lambda=2} = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right\}$$

Let D be a diagonal matrix (a square matrix in which all of the entries are zero, except possibly those on the main diagonal). Then computing powers of D are simple, as this example illustrates:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

Example 5.3.2. If $D = \begin{bmatrix} 5 & 0 \\ 0 & 3 \end{bmatrix}$, then $D^2 = \begin{bmatrix} 5 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 5 & 0 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} 25 & 0 \\ 0 & 9 \end{bmatrix}$

In general, $D^k = \begin{bmatrix} 5^k & 0 \\ 0 & 3^k \end{bmatrix}$.

Similar Matrices

Definition 5.4. Two matrices A and B are said to be **similar** if there is an invertible matrix P such that $B = P^{-1}AP$.

$PB = AP$

If A is not a diagonal matrix itself, but it is similar to a diagonal matrix D, then $A = PDP^{-1}$, for some invertible matrix P. Then

$$A^2 = (PD\overbrace{P^{-1})(P}^{I}DP^{-1}) = PD^2P^{-1}.$$

$$A^3 = (PD\underbrace{P^{-1})(P}_{I}D\underbrace{P^{-1})(P}_{I}DP^{-1}) = PD^3P^{-1}$$

In general, $A^n = PD^nP^{-1}$.

Compare the number of multiplications required to compute each side of this equation when n is very large, and you'll see that this formula can be quite efficient.

Eigenvalue Properties

Theorem 5.2. If A and B are similar they have the same eigenvalues. (Note: This does NOT work in the other direction.)

5.3.2 Diagonalizability

Diagonalizable

Definition 5.5. A square matrix A is said to be **diagonalizable** if $A = PDP^{-1}$ for some diagonal matrix D.

Theorem 5.3. An $n \times n$ matrix A is diagonalizable if and only if A has n linearly independent eigenvectors. In other words, A is diagonalizable if and only if the eigenvectors of A form a basis for $\mathbb{R}^n$.

If this is the case, then the columns of P are the eigenvectors of A, and the diagonal entries of D are the eigenvalues of A, in corresponding order.

Example 5.3.3. Diagnonalize $A = \begin{bmatrix} 3 & 2 & 4 \\ 2 & 0 & 2 \\ 4 & 2 & 3 \end{bmatrix}$. We did the work in example 5.2.6.

$\lambda = -1 \quad \left\{ \begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -2 \\ 1 \end{bmatrix} \right\} = E_{-1}$

$\lambda = 8 \quad \left\{ \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix} \right\} = E_8$

$D = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 8 \end{bmatrix}$ ← eigenvalues

$P = \begin{bmatrix} 1 & 0 & 2 \\ -2 & -2 & 1 \\ 0 & 1 & 2 \end{bmatrix}$ ← eigenvectors

$D_2 = \begin{bmatrix} 8 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$

$P_2 = \begin{bmatrix} 2 & 1 & 0 \\ 1 & -2 & -2 \\ 2 & 0 & 1 \end{bmatrix}$

Example 5.3.4. Diagnonalize $A = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix}$.

$\lambda = 2, 2, 2$

$E_{\lambda=2} = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right\}$

$D = \begin{bmatrix} \; \end{bmatrix}$

$P = [\text{NO}]$

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Not diagonalizable because need 3 eigenvectors

$A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$

$\lambda = 2, 2, 2$

$E_{\lambda=2} \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$

PDP^{-1} $\quad P = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$