Section5.2_Lay6e

Monday, October 31, 2022 2:16 PM

5.2 The Characteristic Equation

5.2.1 The Eigenvalue Problem

λ is a number

We would like to find solutions to

$$A\vec{x} = \lambda\vec{x}$$

where λ is a constant. In these cases multiplication by the matrix is the same as multiplication by a constant. These constants (λ) are called **eigenvalues** and their associated vectors ($\vec{x}$) are called **eigenvectors**.

So how do we find eigenvalues and eigenvectors?

We want $A\vec{x} = \lambda\vec{x}$ so

$$A\vec{x} - \lambda\vec{x} = 0 \quad \leftarrow \text{set equal to zero}$$

$$(A - \lambda I)\vec{x} = 0 \tag{1}$$

$Bx = 0$

$[B \mid 0] \xrightarrow[\text{Reduce}]{\text{Row}}$ Must have row of zeros.

We know this has nonzero solutions if and only if

$$\det(A - \lambda I) = 0 \tag{2}$$

$\operatorname{Det} B = 0$

Step 1: Solve equation (2) for the eigenvalues λ.

Step 2: The eigenvectors are the solutions to equation (1) for a particular value of λ.

Example 5.2.1. Find all eigenvalues and eigenvectors for $A = \begin{bmatrix} 1 & 3 \\ 4 & 2 \end{bmatrix}$ and find a basis for the eigenspace for the solution.

$\{\text{Eigenvectors}\}$ = basis for eigenspace

Step 1: $(A - \lambda I) = \begin{bmatrix} 1 & 3 \\ 4 & 2 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1-\lambda & 3 \\ 4 & 2-\lambda \end{bmatrix}$

subtract λ from the diagonal

$\det(A - \lambda I) = (1 - \lambda)(2 - \lambda) - 12 = 0$

$2 - 3\lambda + \lambda^2 - 12 = 0 \longrightarrow \lambda^2 - 3\lambda - 10 = 0$

$r^2 - 3r - 10 = 0$

$(\lambda - 5)(\lambda + 2) = 0$

$r = 5, -2$

$\boxed{\lambda = 5, -2}$

Step 2: Solve $(A - \lambda I)\vec{x} = 0$ for $\vec{x}$

$\lambda = 5$ solve $(A - 5I)x = 0$

$$\begin{bmatrix} -4 & 3 \\ 4 & -3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\left[\begin{array}{cc|c} -4 & 3 & 0 \\ 4 & -3 & 0 \end{array}\right] \xrightarrow{RR} \left[\begin{array}{cc|c} 1 & -3/4 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

$x_1 - \frac{3}{4}x_2 = 0$

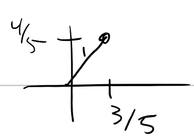
$x_1 = \frac{3}{4}x_2$

$x_2 = x_2$

$E_{\lambda=5} = \left\{ \begin{bmatrix} 3/4 \\ 1 \end{bmatrix} \right\}$

$\boxed{E_{\lambda=5} = \left\{ \begin{bmatrix} 3 \\ 4 \end{bmatrix} \right\}}$

$E_{\lambda=5} = \left\{ \begin{bmatrix} 3/5 \\ 4/5 \end{bmatrix} \right\}$



$\lambda = -2$ solve $(A + 2I)x = 0$

$$\begin{bmatrix} 3 & 3 \\ 4 & 4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\left[\begin{array}{cc|c} 3 & 3 & 0 \\ 4 & 4 & 0 \end{array}\right] \xrightarrow{RR} \left[\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

$x_1 + x_2 = 0$

$x_1 = -x_2$

$x_2 = x_2$

$\boxed{E_{\lambda=-2} = \left\{ \begin{bmatrix} -1 \\ 1 \end{bmatrix} \right\}}$

Characteristic Equation

Definition 5.3. The equation

$$\det(A - \lambda I) = 0$$

equation/polynomial.

is called the **characteristic equation** of the matrix A. A scalar λ is an eigenvalue of an $n \times n$ matrix A if and only if λ is a solution to the characteristic equation of A

Example 5.2.2. Find the characteristic equation of

$$A = \begin{bmatrix} 7 & -2 & -4 & -1 \\ 0 & 1 & 7 & -8 \\ 0 & 0 & 5 & -2 \\ 0 & 0 & 0 & 7 \end{bmatrix}$$

$$\det(A - \lambda I) = \begin{vmatrix} 7-\lambda & -2 & -4 & -1 \\ 0 & 1-\lambda & 7 & -8 \\ 0 & 0 & 5-\lambda & -2 \\ 0 & 0 & 0 & 7-\lambda \end{vmatrix} = \boxed{(7-\lambda)^2(1-\lambda)(5-\lambda)}$$

Note that the factor $(7 - \lambda)$ and the eigenvalue $\lambda = 7$ appear twice. The eigenvalue 7 is said to have multiplicity 2.

Example 5.2.3. Find the characteristic equation, the eigenvalues ($\lambda = 3, 3$) and the eigenvectors of

$$A = \begin{bmatrix} 2 & 1 \\ -1 & 4 \end{bmatrix} \quad \text{solve } (A - \lambda I)x = 0$$

Step 1: $\det(A - \lambda I) = 0$

$$\begin{vmatrix} 2-\lambda & 1 \\ -1 & 4-\lambda \end{vmatrix} = (2-\lambda)(4-\lambda) + 1 = 8 - 2\lambda - 4\lambda + \lambda^2 + 1$$

$$= \lambda^2 - 6\lambda + 9 = (\lambda - 3)^2 = 0$$

$$\boxed{\lambda = 3 \ \& \ \lambda = 3}$$

Step 2: solve $(A - 3I)x = 0$

$$\left[\begin{array}{cc|c} -1 & 1 & 0 \\ -1 & 1 & 0 \end{array}\right] \quad -x_1 + x_2 = 0$$

$$\begin{aligned} x_1 &= x_2 \\ x_2 &= x_2 \end{aligned} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} x_2$$

$$\boxed{E_{\lambda=3} = \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}}$$

Example 5.2.4. Find the characteristic equation, the eigenvalues and the eigenvectors of

$$A = \begin{bmatrix} 3 & -4 \\ 4 & 8 \end{bmatrix}$$

solve $(A-\lambda I)x = 0$

Step 1: $\begin{vmatrix} 3-\lambda & -4 \\ 4 & 8-\lambda \end{vmatrix} = (3-\lambda)(8-\lambda)+16 = 0$

Complex answers § 5.5

Step 2:

Example 5.2.5. Find the characteristic equation, the eigenvalues and the eigenvectors of

$$A = \begin{bmatrix} 4 & 0 & 0 \\ 5 & 3 & 2 \\ -2 & 0 & 2 \end{bmatrix}$$

solve $(A-\lambda I)x = 0$

$\begin{vmatrix} 4-\lambda & 0 & 0 \\ 5 & 3-\lambda & 2 \\ -2 & 0 & 2-\lambda \end{vmatrix} = (4-\lambda)\begin{vmatrix} 3-\lambda & 2 \\ 0 & 2-\lambda \end{vmatrix} = (4-\lambda)(3-\lambda)(2-\lambda) = 0$

$4-\lambda = 0$

$4 = \lambda$

$\lambda = 4, 3, 2$

$\lambda = 4$: $\left[\begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ 5 & -1 & 2 & 0 \\ -2 & 0 & -2 & 0 \end{array}\right] \xrightarrow{RR} \left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & +1 & +3 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$

$x_1 + x_3 = 0$

$+x_2 + 3x_3 = 0$

$x_1 = -x_3$

$x_2 = -3x_3$

$x_3 = x_3$

$\boxed{E_{\lambda=4} = \left\{\begin{bmatrix} -1 \\ -3 \\ 1 \end{bmatrix}\right\}}$

$\lambda = 3$: $\left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 5 & 0 & 2 & 0 \\ -2 & 0 & -1 & 0 \end{array}\right] \xrightarrow{RR} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$

$x_1 = 0$

$x_2 = x_2$

$x_3 = 0$

$\boxed{E_{\lambda=3} = \left\{\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\right\}}$

$\lambda = 2$: $\left[\begin{array}{ccc|c} 2 & 0 & 0 & 0 \\ 5 & 1 & 2 & 0 \\ -2 & 0 & 0 & 0 \end{array}\right] \xrightarrow{RR} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$

$x_1 = 0$

$x_2 = -2x_3$

$x_3 = x_3$

$\boxed{E_{\lambda=2} = \left\{\begin{bmatrix} 0 \\ -2 \\ 1 \end{bmatrix}\right\}}$

5.2.2 Eigenvalue Properties

Eigenvalue Properties

Some properties of eigenvalues and eigenvectors (Eigenpairs)

1. Eigenvectors are not unique.
2. A matrix can have a zero eigenvalue
3. A real matrix may have one or more complex eigenvalues and eigenvectors.
4. Eigenvectors corresponding to distinct eigenvalues are linearly independent.

$$\det\begin{bmatrix} 1-\lambda & 1 \\ 1 & 1-\lambda \end{bmatrix}$$

$$(1-\lambda)^2-1=0$$

$$\cancel{1}-2\lambda+\lambda^2\cancel{-1}=0$$

$$\lambda(\lambda-2)=0$$

$$\lambda=0 \quad \lambda=2$$

The Invertible Matrix Theorem (continued)

The Invertible Matrix Theorem (continued)
Let A be an $n\times n$ matrix. Then the following statements are equivalent to the statements found in the Invertible Matrix Theorem given in Chapter 2 (including the statement that A is invertible):

s. The number 0 is *not* an eigenvalue of A.

t. The determinant of A is *not* zero.

Example 5.2.6. Find the eigenvalues, eigenvectors and a basis for the eigenspace for

$$A=\begin{bmatrix} 3 & 2 & 4 \\ 2 & 0 & 2 \\ 4 & 2 & 3 \end{bmatrix}$$ (Partial solution: Basis for $E_{-1}=\{(1,-4,1),(1,0,-1)\}$)

$$0=\begin{vmatrix} 3-\lambda & 2 & 4 \\ 2 & -\lambda & 2 \\ 4 & 2 & 3-\lambda \end{vmatrix}=(3-\lambda)\begin{vmatrix} -\lambda & 2 \\ 2 & 3-\lambda \end{vmatrix}-2\begin{vmatrix} 2 & 2 \\ 4 & 3-\lambda \end{vmatrix}+4\begin{vmatrix} 2 & -\lambda \\ 4 & 2 \end{vmatrix}$$

$$=(3-\lambda)\left[-\lambda(3-\lambda)-4\right]-2\left[2(3-\lambda)-8\right]+4\left[4+4\lambda\right]$$

$$=(3-\lambda)\left[\lambda^2-3\lambda-4\right]-2\left[-2-2\lambda\right]+4\left[4+4\lambda\right]$$

$$=(3-\lambda)(\lambda-4)(\lambda+1)-2(-2)(\lambda+1)+4(4)(\lambda+1)$$

$$=(\lambda+1)\left[(3-\lambda)(\lambda-4)+4+16\right]$$

$$=(\lambda+1)\left[3\lambda-\lambda^2-12+4\lambda+20\right]$$

$$=(\lambda+1)\left[\lambda^2-7\lambda-8\right](-1)$$

$$=(\lambda+1)(\lambda+1)(\lambda-8)$$

$$\lambda=-1,\ -1,\ 8$$