Section5.1_Lay6e

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5.1 Eigenvectors and Eigenvalues

For this section we will assume that A is an $n \times n$ matrix. So any transformation $T(x) = Ax$ sends vectors from $\mathbb{R}^n$ to $\mathbb{R}^n$.

Example 5.1.1. Consider the linear transformation $T(x) = Ax$ defined by $A = \begin{bmatrix} 4 & 1 \\ -2 & 1 \end{bmatrix}$.

1. Describe what this transformaton does to the standard basis vectors in $\mathbb{R}^2$

$$A\begin{bmatrix}1\\0\end{bmatrix} = \begin{bmatrix}4 & 1\\-2 & 1\end{bmatrix}\begin{bmatrix}1\\0\end{bmatrix} = \begin{bmatrix}4\\-2\end{bmatrix} \qquad A\begin{bmatrix}0\\1\end{bmatrix} = \begin{bmatrix}4 & 1\\-2 & 1\end{bmatrix}\begin{bmatrix}0\\1\end{bmatrix} = \begin{bmatrix}1\\1\end{bmatrix}$$

2. Let $\mathfrak{b}_1 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$. Calculate $T(\mathfrak{b}_1)$ and describe what happens.

$$\begin{bmatrix}4 & 1\\-2 & 1\end{bmatrix}\begin{bmatrix}-1\\1\end{bmatrix} = \begin{bmatrix}-4+1\\2+1\end{bmatrix} = \begin{bmatrix}-3\\3\end{bmatrix} = 3\begin{bmatrix}-1\\1\end{bmatrix} = 3b_1$$

eigenvector

eigenvalue

3. Let $\mathfrak{b}_2 = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$. Calculate $T(\mathfrak{b}_2)$ and describe what happens.

$$\begin{bmatrix}4 & 1\\-2 & 1\end{bmatrix}\begin{bmatrix}-1\\2\end{bmatrix} = \begin{bmatrix}-4+2\\2+2\end{bmatrix} = \begin{bmatrix}-2\\4\end{bmatrix} = 2\begin{bmatrix}-1\\2\end{bmatrix} = 2b_2$$

eigenvalue eigenvector

$$A\begin{bmatrix}-2\\4\end{bmatrix} = \begin{bmatrix}-4\\8\end{bmatrix}$$

Eigenvectors and Eigenvalues

Definition 5.1. An **eigenvector** of an $n \times n$ matrix A is a nonzero vector x such that $Ax = \lambda x$ for some scalar λ. The scalar λ is called an **eigenvalue** of A if there is a nontrivial solution to the equation $Ax = \lambda x$. (Note that an eigenvector must be nonzero but eignevalues can be zero.)

λ is unique
x is not

In this section we will be given either an eigenvalue or an eigenvector for each problem.

$Ax - \lambda I x = 0$ can't be done this way

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$(A - \lambda I)x = 0$

show this

Example 5.1.2. Let $A = \begin{bmatrix} 3 & 2 \\ 3 & 8 \end{bmatrix}$. Show that $b = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$ is an eigenvector of A. Determine the corresponding eigenvalue.

Eigenvector

$$\boxed{Ab = \lambda b}$$

λ is the eigenvalue

$$\begin{bmatrix} 3 & 2 \\ 3 & 8 \end{bmatrix}\begin{bmatrix} 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 9 \\ 27 \end{bmatrix} = 9\begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

$$A \quad b = 9 \quad b$$

$\boxed{\lambda = 9 \text{ is the eigenvalue}}$

solve for b

Example 5.1.3. Let $A = \begin{bmatrix} 3 & 2 \\ 3 & 8 \end{bmatrix}$. Show that $\lambda = 2$ is an eigenvalue of A. Determine eigenvector whose eigenvalue is 2.

$$b = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

$$Ab = 2b$$

$$\begin{bmatrix} 3 & 2 \\ 3 & 8 \end{bmatrix}\begin{bmatrix} b_1 \\ b_2 \end{bmatrix} = 2\begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

$$\begin{cases} 3b_1 + 2b_2 = 2b_1 \\ 3b_1 + 8b_2 = 2b_2 \end{cases}$$

solve

$$\begin{cases} b_1 + 2b_2 = 0 \\ 3b_1 + 6b_2 = 0 \end{cases} \rightarrow \begin{bmatrix} 1 & 2 & 0 \\ 3 & 6 & 0 \end{bmatrix} \xrightarrow{RR}$$

$$\xrightarrow{RR} \begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$b_1 + 2b_2 = 0$

$b_1 = -2b_2$

$b_2 = b_2$

$$\begin{bmatrix} 3b_1 + 2b_2 \\ 3b_1 + 8b_2 \end{bmatrix} = \begin{bmatrix} 2b_1 \\ 2b_2 \end{bmatrix}$$

$$Ab - 2Ib = 0$$

$$\begin{bmatrix} 3 & 2 \\ 3 & 8 \end{bmatrix} b - 2\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} b = 0 \qquad \begin{bmatrix} 3-2 & 2 \\ 3 & 8-2 \end{bmatrix} b = 0$$

$$\boxed{b = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \end{bmatrix} b_2}$$

Eigenspace

Definition 5.2. The set of all solutions to $(A - \lambda I)x = 0$ is a subspace of $\mathbb{R}^n$. It is called the **eigenspace** of A corresponding to the eigenvalue λ.

Example 5.1.4. Let $A = \begin{bmatrix} 6 & 3 & -4 \\ 2 & 7 & -4 \\ 2 & 3 & 0 \end{bmatrix}$. if $\lambda = 4$ is an eigenvalue of A find a basis for the eigenspace of A

Solve $(A - \lambda I)x = 0$

$$\left(\begin{bmatrix} 6 & 3 & -4 \\ 2 & 7 & -4 \\ 2 & 3 & 0 \end{bmatrix} - 4\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right)\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$$\begin{bmatrix} 2 & 3 & -4 \\ 2 & 3 & -4 \\ 2 & 3 & -4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \longrightarrow \begin{bmatrix} 2 & 3 & -4 & 0 \\ 2 & 3 & -4 & 0 \\ 2 & 3 & -4 & 0 \end{bmatrix} \xrightarrow{RR} \begin{bmatrix} 1 & 3/2 & -2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$x_1 + \frac{3}{2}x_2 - 2x_3 = 0$

$x_1 = -\frac{3}{2}x_2 + 2x_3$

$x_2 = x_2$

$x_3 = x_3$

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -3/2 \\ 1 \\ 0 \end{bmatrix} x_2 + \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} x_3$$

Basis for $\lambda = 4$ = $\left\{ \begin{bmatrix} -3/2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} \right\}$

$E_{\lambda=4}$

Eigenvector Linear Independence Theorem

Theorem 5.1. If $v_1, \ldots, v_r$ are eigenvectors that correspond to distinct eigenvalues $\lambda_1, \ldots, \lambda_r$ of an $n \times n$ matrix A, then the set $\{v_1, \ldots, v_r\}$ is linearly independent.

$x_1 = x_1$

$x_2 = x_2$

$x_3 = \frac{1}{2}x_1 + \frac{3}{4}x_2$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 1/2 \end{bmatrix} x_1 + \begin{bmatrix} 0 \\ 1 \\ 3/4 \end{bmatrix} x_2$$

Example 5.1.5. A is an $m \times n$ matrix. Mark each statement TRUE or FALSE. $x \neq 0$

Row reduce $A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

1. If $Ax = \lambda x$ for some vector x, then λ is an eigenvalue of A.

   True

2. A matrix A is not invertible if and only if 0 is an eigenvalue of A ($Ax = 0x$).

   True

3. A number c is an eigenvalue of A if and only if the equation $(A - cI)x = 0$ has a nontrivial solution.

   $x = 0$ is NEVER an eigenvector

4. To find the eigenvalues of A, reduce A to echelon form.

   NO solve $(A - \lambda I)x = 0$

5. If $Ax = \lambda x$ for some scalar λ, then x is an eigenvector of A.

   True

6. If v_1 and v_2 are linearly independent eigenvectors, then they correspond to distinct eigenvalues.

   False see ex 5.1.4

7. An eigenspace of A is a null space of a certain matrix.

   True $(\text{Matrix})x = 0$ is null space

   $(A - \lambda I)x = 0$

8. An $n \times n$ matrix can have at most n eigenvalues.

   True