Section 4.5

Monday, October 3, 2022 1:44 PM

$$\begin{bmatrix} 1 & 4 & 0 & 2 & -1 \\ 3 & 12 & 1 & 5 & 5 \\ 2 & 8 & 1 & 3 & 2 \\ 5 & 20 & 2 & 8 & 8 \end{bmatrix} \xrightarrow{\text{RREF}} \begin{array}{c} \left[\begin{array}{ccccc|c} 1 & 4 & 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right] \\ \begin{array}{ccccc} x_1 & x_2 & x_3 & x_4 & x_5 \end{array} \end{array}$$

4.5 Dimension of a Vector Space

4.5.1 Dimension

Dimension

Definition 4.9. Let S be a subspace of $\mathbb{R}^n$ for some n, and $\mathcal{B}$ be a basis for S. The **dimension** of S is the number of vectors in $\mathcal{B}$.

Example 4.5.1. Find the dimensions of the null space and the column space of

5 variables / Vectors

$$B = \begin{bmatrix} 1 & 4 & 0 & 2 & -1 \\ 3 & 12 & 1 & 5 & 5 \\ 2 & 8 & 1 & 3 & 2 \\ 5 & 20 & 2 & 8 & 8 \end{bmatrix} \xrightarrow{RR} \begin{array}{c} \begin{array}{ccccc} x_1 & x_2 & x_3 & x_4 & x_5 \end{array} \\ \left[\begin{array}{ccccc|c} 1 & 4 & 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right] \end{array}$$

Free variables

$x_1 + 4x_2 + 2x_4 = 0$
$x_3 - x_4 = 0$
$x_5 = 0$

$x_1 = -4x_2 - 2x_4$
$x_2 = x_2$
$x_3 = x_4$
$x_4 = x_4$
$x_5 = 0$

$$\text{Nul } B = \left\{ \begin{bmatrix} -4 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ 1 \\ 1 \\ 0 \end{bmatrix} \right\}$$

dim Nul B = 2

$$\text{Col } B = \left\{ \begin{bmatrix} 1 \\ 3 \\ 2 \\ 5 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} -1 \\ 5 \\ 2 \\ 8 \end{bmatrix} \right\}$$

dim Col B = 3

Example 4.5.2. Find the dimensions of the null space and the column space of

$$A = \begin{bmatrix} 3 & 6 & 1 & 1 & 7 \\ 1 & 2 & 2 & 3 & 1 \\ 2 & 4 & 5 & 8 & 4 \end{bmatrix}.$$

(You do not have to find the basis vectors, just the dimensions.)

Pivot columns = col space
Free variables = null space

$$\begin{bmatrix} 1 & 2 & 2 & 3 & 1 \\ 3 & 6 & 1 & 1 & 7 \\ 2 & 4 & 5 & 8 & 4 \end{bmatrix} \xrightarrow[R_3 - 2R_1]{R_2 - 3R_1} \begin{bmatrix} 1 & 2 & 2 & 3 & 1 \\ 0 & 0 & -5 & -8 & 4 \\ 0 & 0 & 1 & 2 & 2 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & 2 & 3 & 1 \\ 0 & 0 & 1 & 2 & 2 \\ 0 & 0 & -5 & -8 & 4 \end{bmatrix} \xrightarrow{\frac{1}{2}(R_3 + 5R_2)} \begin{bmatrix} 1 & 2 & 2 & 3 & 1 \\ 0 & 0 & 1 & 2 & 2 \\ 0 & 0 & 0 & 1 & 7 \end{bmatrix}$$

free variables

$$\text{Col } \left\{ \begin{bmatrix} 1 \\ 0 \\ \vdots \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ \vdots \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$

Dim Nul = 2
dim col = 3

Col $\left\{ \begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\end{bmatrix} \right\}$

Dim Nul = 2

dim Col = 3

Example 4.5.3. Find a basis and state the dimension of the subspace: $\left\{ \begin{bmatrix} a+b \\ 2a \\ 3a-b \\ -b \end{bmatrix} : a, b \in \mathbb{R} \right\} = B$

$$B = \left\{ \begin{bmatrix}1\\2\\3\\0\end{bmatrix}, \begin{bmatrix}1\\0\\-1\\-1\end{bmatrix} \right\}$$

$$\boxed{\dim = 2}$$

$$\begin{bmatrix} a+b \\ 2a \\ 3a-b \\ -b \end{bmatrix} = \begin{bmatrix} a \\ 2a \\ 3a \\ 0 \end{bmatrix} + \begin{bmatrix} b \\ 0 \\ -b \\ -b \end{bmatrix}$$

a = 1 b = 1

Example 4.5.4. V is vector space. Mark each statement TRUE or FALSE.

1. The number of pivot columns of a matrix equals the dimension of its column space.

   True

2. a plane in $\mathbb{R}^3$ is a two-dimensional subspace of $\mathbb{R}^3$.

   True plane is 2-D Needs 2 vectors to describe.

3. If dim $V = n$ and S is a linearly independent set in V, then S is a basis for V.

   False S is a basis for a SUBSPACE of V.

4. If a set $\{v_1, \ldots, v_p\}$ spans a finite-dimensional vector space V and if T is a set of more than p vectors in V, then T must be linearly dependent.

   False $V = \left\{ \begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix} \right\}$ $T = \left\{ \begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\end{bmatrix} \right\}$

5. $\mathbb{R}^2$ is a two-dimensional subspace of $\mathbb{R}^3$.

   False $\mathbb{R}^2$ is not in $\mathbb{R}^3$.

6. The number of variables in the equation $ax = 0$ equals the dimension of Nul A.

   False #of FREE variables.

7. A vector space is infinite-dimensional if it is spanned by an infinite set.

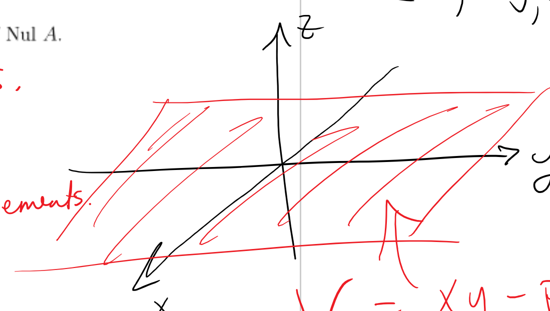
   T

8. If dim $V = n$ and if S spans V, then S is a basis of V.

   False would be basis IF n elements.

9. The only three-dimensional subspace of $\mathbb{R}^3$ is $\mathbb{R}^3$ itself.

   TRUE



V = xy-plane

4.5.2 Rank (dimension of Col A) and Nullity (Dimension of Nul A)

Rank

Definition 4.10. The **rank** of a matrix A is the dimension of the column space of A. The **nullity** of A is the dimension of the null space of A.

Theorem 4.5 (The Rank Theorem). If A is an $m \times n$ matrix then

$$\text{rank } A + \text{nullity } A = n.$$

\# of free variables

pivot columns

Example 4.5.5. Answer the following about rank of matrices.

(a) Can a 6×9 matrix have a two-dimensional null space? NO

6 rows 9 columns at most 6 pivots

a b c

9 colums

at least 3 free variables

(b) What is the minimum rank of a 5×7 matrix?

1

(c) What is the maximum rank of a 5×7 matrix?

5

(d) What is the minimum rank of a 7×5 matrix?

1

(e) What is the maximum rank of a 7×5 matrix?

5 7 rows 5 columns

(f) If a 7×7 matrix is invertible, what is its rank?

7 b/c can reduce to I_7

(g) If the subspace of all solution of $Ax = 0$ has a basis consisting of three vectors and if A is a 5×7 matrix, what is the rank of A?

7 columns

Rank A + Nul A = 7

Rank A + 3 = 7

$\boxed{\text{Rank } A = 4}$

(h) What is the rank of a 4×5 matrix whose null space is three-dimensional?

5 columns

Rank A + 3 = 5

$\boxed{\text{Rank } A = 2}$

(i) If the rank of a 7×6 matrix A is 4, what is the dimension of the solution space of $Ax = 0$?

solution to homogeneous is null space

6 columns

Rank A + Nul A = 6

4 + Nul A = 6

$\boxed{\text{Nul } A = 2}$

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Invertible Matrix Theorem (continued)

Theorem [The Invertible Matrix Theorem (continued)]
Let A be an $n \times n$ matrix. Then the following statements are equivalent to the statements found in the Invertible Matrix Theorem given in Chapter 2 (including the statement that A is invertible):

m. The columns of A form a basis for $\mathbb{R}^n$.

n. Col $A = \mathbb{R}^n$.

o. dim Col $A = n$.

$\boxed{\text{Nul}A = 2}$

Invertible Matrix Theorem (continued)

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m. The columns of A form a basis for $\mathbb{R}^n$.

n. Col $A = \mathbb{R}^n$.

o. dim Col $A = n$.

p. rank $A = n$.

q. Nul $A = 0$.

r. dim Nul $A = 0$.

Example 4.5.6. A is an $m \times n$ matrix. Mark each statement TRUE or FALSE.

1. The row space of A is the same as the column space of A^T.

T

vectors of size m

Also vectors of size m and the same number of them.

2. If B is any echelon form of A, and if B has three nonzero rows, then the first three rows of A form a basis for Row A.

False

$$\begin{bmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \\ 1 & 3 & 4 \end{bmatrix} \xrightarrow{RR} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 3 \\ 0 & 0 & 0 \end{bmatrix}$$

3. The dimensions of the row space and the column space of A are the same, even if A is not square.

True pivot rows → pivot columns

4. If B is any echelon form of A, then the pivot columns of B form a basis for the column space of A.

False Have to use A

5. Row operations preserve the linear dependence relations among the rows of A.

False see #2

6. The dimension of the null space of A is the number of columns of A that are *not* pivot columns.

free variables

True

7. The row space of A^T is the same as the coulumn space of A.

True

8. If A and B are row equivalent, then their row spaces are the same.

True

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \\ 1 & 3 & 4 \end{bmatrix} \sim B = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 3 \\ 0 & 0 & 0 \end{bmatrix}$$

same row space