4.4 Coordinate Systems

Some basis theorems

Theorem 4.3. If a vector space V has a basis $\mathcal{B} = \{b_1, \ldots, b_n\}$, then any set in V containing more than n vectors must be linearly dependent.

Theorem 4.4 (The Unique Representation Theorem)**.** Any vector x in vector space V can be written in only one way as a linear combination of basis vectors.

Definition 4.8. Suppose the set $\mathcal{B} = \{b_1, \ldots, b_n\}$ is an ordered basis for a subspace H. For each x in H, the **coordinate of** x **relative to the basis** $\mathcal{B}$ are the weights $c_1, \ldots, c_n$ such that $x = c_1b_1 + \cdots + c_nb_n$, and the vector in $\mathbb{R}^n$

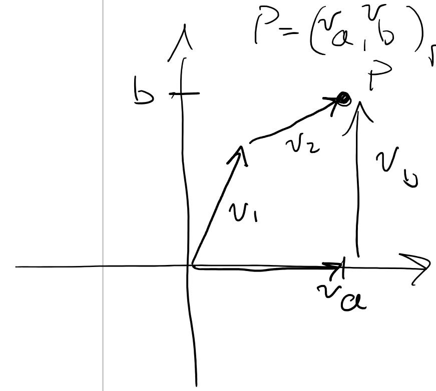
$c_2b_2 + \cdots + c_nb_n$

$$[x]_{\mathcal{B}} = \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}_B$$

is called the **coordinate vector of** x **(relative to** $\mathcal{B}$**)** or the $\mathcal{B}$**-coordinate vector of** x

$P = (v_1, v_2)_{B_2}$

$P = (v_a, v_b)_{B_1}$



$\mathbb{R}^2 = \left\{ \begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix} \right\} = \{x, y\}$

$\mathbb{R}^2 = \left\{ \begin{pmatrix}0\\1\end{pmatrix}, \begin{pmatrix}1\\0\end{pmatrix} \right\} = \{y, x\}$

Example 4.4.1. Converting from the alternate basis to the standard basis. Suppose

$$\mathcal{B} = \left\{ \begin{bmatrix}1\\2\\0\end{bmatrix}, \begin{bmatrix}3\\1\\1\end{bmatrix}, \begin{bmatrix}-1\\0\\-1\end{bmatrix} \right\} \text{ and } [u]_{\mathcal{B}} = \begin{bmatrix}4\\-1\\2\end{bmatrix}_B.$$ Find u in the standard basis.

$\{b_1, b_2, b_3\}$

$$[u]_B = \begin{bmatrix}4\\-1\\2\end{bmatrix}_B = 4b_1 + (-1)b_2 + 2b_3 = 4\begin{bmatrix}1\\2\\0\end{bmatrix} - \begin{bmatrix}3\\1\\1\end{bmatrix} + 2\begin{bmatrix}-1\\0\\-1\end{bmatrix}$$

$$= \begin{bmatrix}4-3-2\\8-1+0\\0-1-2\end{bmatrix} = \begin{bmatrix}-1\\7\\-3\end{bmatrix}$$

standard basis

$\left\{ \begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\end{pmatrix}, \begin{pmatrix}0\\0\\1\end{pmatrix} \right\} = \{e_1, e_2, e_3\}$

$P(a,b) = a\begin{pmatrix}1\\0\end{pmatrix} + b\begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}a\\b\end{pmatrix}$

$a b_1 + b b_2$

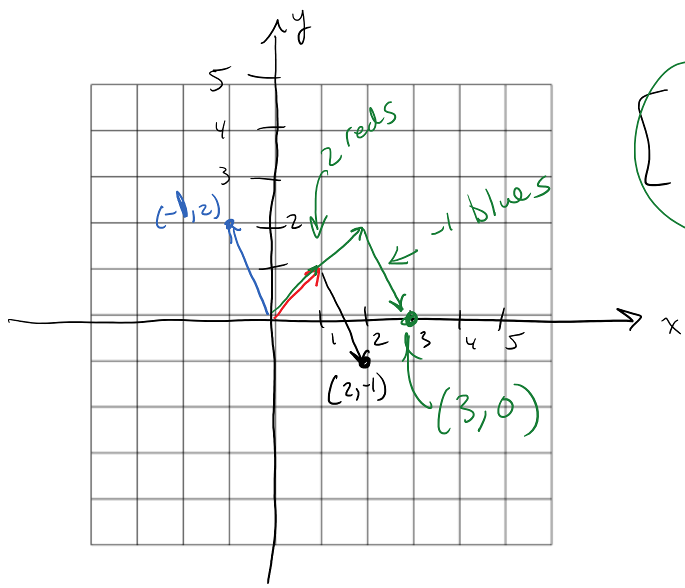
Example 4.4.2. $\mathcal{B} = \left\{ \begin{bmatrix}1\\1\end{bmatrix}, \begin{bmatrix}-1\\2\end{bmatrix} \right\} = \{b_1, b_2\}$ is a basis for $\mathbb{R}^2$. The standard basis for $\mathbb{R}^2$ is $\mathcal{E} = \{e_1, e_2\}$. Write the point P that is called $\begin{bmatrix}2\\-1\end{bmatrix}_B$ in the ~~standard~~ basis in terms of basis ~~$\mathcal{B}$~~ $\mathcal{E}$

epsilon

$\{e_1, e_2\} = \left\{ \begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix} \right\}$

Notation: $Q = \begin{bmatrix}2\\-3\end{bmatrix}_{\mathcal{B}} = (2)b_1 + (-3)b_2 = (2)\begin{bmatrix}1\\1\end{bmatrix} + (-3)\begin{bmatrix}-1\\2\end{bmatrix} = \begin{bmatrix}5\\-4\end{bmatrix}_{\mathcal{E}}$

standard basis



$$\begin{bmatrix}2\\-1\end{bmatrix}_B = 2\begin{bmatrix}1\\1\end{bmatrix} - 1\begin{bmatrix}-1\\2\end{bmatrix} = \boxed{\begin{bmatrix}3\\0\end{bmatrix}_{\mathcal{E}}}$$

$\begin{bmatrix}1\\1\end{bmatrix}$ $\begin{bmatrix}-1\\2\end{bmatrix}$

$\begin{bmatrix}2\\-1\end{bmatrix}_B$

answer.

actual problem

$a=1$ $b=-1$

$$\begin{bmatrix}2\\-1\end{bmatrix}_{\mathcal{E}} = \begin{bmatrix}a\\b\end{bmatrix}_B = a\begin{bmatrix}1\\1\end{bmatrix} + b\begin{bmatrix}-1\\2\end{bmatrix} = \begin{bmatrix}2\\-1\end{bmatrix}_{\mathcal{E}}$$

← solve for a & b

$$\begin{bmatrix}1 & -1 & 2\\1 & 2 & -1\end{bmatrix} \xrightarrow{R_2 - R_1} \begin{bmatrix}1 & -1 & 2\\0 & 3 & -3\end{bmatrix} \xrightarrow[R_1+R_2]{\frac{1}{3}R_2} \begin{bmatrix}1 & 0 & 1\\0 & 1 & -1\end{bmatrix}$$

$$\boxed{\begin{bmatrix}a\\b\end{bmatrix}_B = \begin{bmatrix}1\\-1\end{bmatrix}_B}$$

actual answer. →

Example 4.4.3. Converting from the standard basis to an alternate basis. Suppose $\mathcal{B} = \left\{ \begin{bmatrix}1\\3\end{bmatrix}, \begin{bmatrix}-1\\1\end{bmatrix} \right\}$ and $u = \begin{bmatrix}4\\8\end{bmatrix}_{\mathcal{E}}$. Find $[u]_{\mathcal{B}}$.

$$\begin{bmatrix}4\\8\end{bmatrix}_{\mathcal{E}} = a\begin{bmatrix}1\\3\end{bmatrix} + b\begin{bmatrix}-1\\1\end{bmatrix} = \begin{bmatrix}a\\b\end{bmatrix}_B$$

$$\left[\begin{array}{cc|c}1 & -1 & 4\\3 & 1 & 8\end{array}\right] \xrightarrow{RR} \begin{bmatrix}1 & 0 & 3\\0 & 1 & -1\end{bmatrix}$$

$$\begin{pmatrix}3\\-1\end{pmatrix} = \begin{pmatrix}a\\b\end{pmatrix}$$

$$\left[\begin{array}{cc|c} 1 & -1 & 4 \\ 3 & 1 & 8 \end{array}\right] \xrightarrow{RR} \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & -1 \end{bmatrix}$$

$$\begin{array}{cc} \uparrow & \uparrow \\ a & b \end{array}$$

$$\boxed{\begin{bmatrix} 3 \\ -1 \end{bmatrix}_B = \begin{bmatrix} a \\ b \end{bmatrix}_B}$$

Example 4.4.4. Converting from the standard basis to a basis for a subspace. Suppose $\mathcal{B} = \left\{ \begin{bmatrix} 3 \\ 6 \\ 2 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \right\}$ and $x = \begin{bmatrix} 3 \\ 12 \\ 7 \end{bmatrix}_{\mathcal{E}}$. Determine if x is in the plane spanned by $\mathcal{B}$, and if so, find $[x]_{\mathcal{B}}$

$$[X]_B = \begin{bmatrix} a \\ b \end{bmatrix}_B = a \begin{bmatrix} 3 \\ 6 \\ 2 \end{bmatrix} + b \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \overset{?}{=} \begin{bmatrix} 3 \\ 12 \\ 7 \end{bmatrix}$$

$$\left[\begin{array}{cc|c} 3 & -1 & 3 \\ 6 & 0 & 12 \\ 2 & 1 & 7 \end{array}\right] \to \text{Row Reduce}$$

$$\begin{bmatrix} 1 & 0 & 2 \\ 3 & -1 & 3 \\ 2 & 1 & 7 \end{bmatrix} \xrightarrow[R_3 - 2R_1]{R_2 - 3R_1} \begin{bmatrix} 1 & 0 & 2 \\ 0 & -1 & -3 \\ 0 & 1 & 3 \end{bmatrix} \checkmark$$

$$\boxed{a = 2 \qquad b = 3 \quad \checkmark}$$