Section 4.2

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Section4.2_Lay6e

$7x = 0$

$x = 0$

4.2 Null Spaces and Column Spaces

Recall:

A **subspace** H of a vector space V is a set of vectors in V which is closed under addition and scalar multiplication.

Example 4.2.1. Is the set of solutions to the matrix equation $Ax = 0$ for $x \in \mathbb{R}^n$ a vector space?

Solution: zero vector? Yes, $x = 0$ is a solution to the equation.
Closed? Suppose u and v are solutions to $Ax = 0$. What about $u + v$?

$Au = 0$

$Av = 0$

$$\begin{aligned} A(u+v) &= Au + Av \\ &= 0 + 0 \end{aligned}$$

Therefore closed.
The set of solutions to the equation $Ax = 0$ is a subspace of $\mathbb{R}^n$ called the null space of matrix A

Null Space

Definition 4.4. The **null space** of a matrix A (Nul A) is the span of the vectors obtained by solving $Ax = 0$.

$$\text{Nul } A = \{x : \ x \text{ is in } \mathbb{R}^n \text{ and } Ax = 0\}$$

All solutions to homogeneous equation.

Example 4.2.2. Suppose the reduced echelon form for a matrix A is $\begin{array}{ccccc} x_1 & x_2 & x_3 & x_4 & x_5 \end{array}$ $\left[\begin{array}{ccccc|c} 1 & 3 & 0 & -1 & 8 & 0 \\ 0 & 0 & 1 & -2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$. Find the solution set of the equation $Ax = 0$. (ie. Find Nul A)

$x_1 + 3x_2 - x_4 + 8x_5 = 0$

$x_3 - 2x_4 + x_5 = 0$

$$\left.\begin{aligned} x_1 &= -3x_2 + x_4 - 8x_5 \\ x_2 &= x_2 \\ x_3 &= 2x_4 - x_5 \\ x_4 &= x_4 \\ x_5 &= x_5 \end{aligned}\right\} \text{Solution to } Ax = 0$$

$$\underbrace{x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} x_2 + \begin{bmatrix} 1 \\ 0 \\ 2 \\ 1 \\ 0 \end{bmatrix} x_4 + \begin{bmatrix} -8 \\ 0 \\ -1 \\ 0 \\ 1 \end{bmatrix} x_5}_{\text{solution to } Ax=0}$$

$$\boxed{\text{Nul } A = \left\{ \begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -8 \\ 0 \\ -1 \\ 0 \\ 1 \end{bmatrix} \right\}}$$

$x \mapsto Ax$

$\mathbb{R}^3 \to \mathbb{R}^2$

Example 4.2.3. Consider the linear transformation $T(x) = Ax$ where $A = \begin{bmatrix} 1 & 2 & -1 \\ 2 & 5 & 0 \end{bmatrix}$.

$\mathbb{R}^3$ $\mathbb{R}^2$

$$\begin{bmatrix} 1 & 2 & -1 \\ 2 & 5 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 1+6-1 \\ 2+15 \end{bmatrix} = \begin{bmatrix} 6 \\ 17 \end{bmatrix} \text{ NO}$$

1. Is the vector $x = \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix}$ in Nul A? Is $A\begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix} = 0$?

1. Is the vector $x = \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix}$ in Nul A? Is $A\begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix} = 0$?

2. Find a description of Nul A. $[A|0]$ RR

$$\left[\begin{array}{ccc|c} 1 & 2 & -1 & 0 \\ 2 & 5 & 0 & 0 \end{array}\right] \xrightarrow{R_2-2R_1} \left[\begin{array}{ccc|c} 1 & 2 & -1 & 0 \\ 0 & 1 & 2 & 0 \end{array}\right] \xrightarrow{R_1-2R_2} \begin{array}{c} \begin{array}{ccc} x_1 & x_2 & x_3 \end{array} \\ \left[\begin{array}{ccc|c} 1 & 0 & -5 & 0 \\ 0 & 1 & 2 & 0 \end{array}\right] \end{array}$$

$x_1 - 5x_3 = 0$
$x_2 + 2x_3 = 0$

$x_1 = 5x_3$
$x_2 = -2x_3$
$x_3 = x_3$

$$X = \begin{bmatrix} 5 \\ -2 \\ 1 \end{bmatrix} x_3$$

$$\boxed{\text{Nul } A = \left\{ \begin{bmatrix} 5 \\ -2 \\ 1 \end{bmatrix} \right\}}$$

1st two columns

Column Spaces

Definition 4.5. The **column space** of a matrix $A = [a_1\ a_2\ \cdots a_n]$ (Col A) is the span of the columns of A.

$$\text{Col } A = \text{Span } \{a_1,\ a_2,\ \ldots, a_n\}$$
$$\text{Col } A = \{b : b = Ax \text{ for some } x \in \mathbb{R}^n\}$$

Example 4.2.4. Consider the matrix $A = \begin{bmatrix} 1 & 2 & -1 \\ 2 & 5 & 0 \end{bmatrix}$.

1. What is Col A?

2. What is the smallest representation of Col A?

#1 $\boxed{\text{Col } A = \left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 5 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \end{bmatrix} \right\}}$

#2 $\boxed{\text{Col } A = \left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 5 \end{bmatrix} \right\}}$

Smallest representation of Col A is the pivot columns in the ORIGINAL matrix.

6

$m \times n$ = rows x columns

Example 4.2.5. A is an $m \times n$ matrix. Mark each statement TRUE or FALSE.

T 1. The null space of A is the solution set of the equation $Ax = 0$.

F 2. The null space of an $m \times n$ matrix is in $\mathbb{R}^m$ null space in $\mathbb{R}^n$

$$\begin{bmatrix} a_{11} & \cdots & a_{1n} \\ a_{21} & & \vdots \\ \vdots & & \\ a_{m1} & & a_{mn} \end{bmatrix} \begin{bmatrix} b_1 \\ \vdots \\ \vdots \\ b_n \end{bmatrix}$$

T 3. The column space of A is the range of the mapping $x \mapsto Ax$.

T 4. Col A is the set of all vectors that can be written as Ax for some x.

T 5. A null space is a vector space.

T 6. The column space of an $m \times n$ matrix is in $\mathbb{R}^m$.

F 7. Col A is the set of all solutions of $Ax = b$. Col A span of columns of A.

Ax = all linear combinations of the columns of A.

b

T 8. The range of a linear transformation is a vector space.

Example 4.2.6. Let

$$A = \begin{array}{c} \begin{array}{ccccc} x_1 & x_2 & x_3 & x_4 & x_5 \end{array} \\ \begin{bmatrix} 1 & 4 & 2 & 6 & 8 \\ 1 & 3 & 2 & -2 & 1 \\ 2 & 0 & 1 & 5 & 2 \end{bmatrix} \end{array}.$$

Find Nul A, Col A and the simplest form of Col A. The reduced echelon form of A is

$$\left[\begin{array}{ccccc|c} 1 & 0 & 0 & 12 & 8 & 0 \\ 0 & 1 & 0 & 8 & 7 & 0 \\ 0 & 0 & 1 & -19 & -14 & 0 \end{array}\right]$$

because spans $\mathbb{R}^3$

$$\text{Col } A = \left\{ \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 4 \\ 3 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix} \right\} = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$

$x_1 = -12x_4 - 8x_5$

$x_2 = -8x_4 - 7x_5$

$x_3 = 19x_4 + 14x_5$

$x_4 = x_4$

$x_5 = x_5$

$$\text{Nul } A = \left\{ \underset{x_4}{\begin{bmatrix} -12 \\ -8 \\ 19 \\ 1 \\ 0 \end{bmatrix}}, \underset{x_5}{\begin{bmatrix} -8 \\ -7 \\ 14 \\ 0 \\ 1 \end{bmatrix}} \right\}$$

Ex: A is $n \times n$ matrix

If Col A = Nul A

Show Nul $A^2 = \mathbb{R}^n$

Pf: Let $x \in \mathbb{R}^n$

then $Ax \in$ Col A

Col A = Nul A so

$\in$ Is an element of

If $y \in$ Nul A

then $Ay = 0$

$\operatorname{Col} A = \operatorname{Nul} A$ so

$Ax \in \operatorname{Nul} A$

so $A(Ax) = 0$

$A^2 x = 0$

so $x \in \operatorname{Nul} A^2$

then $Ay = 0$

so $\operatorname{Nul} A^2 = \mathbb{R}^n$

Row space is the span of the rows of A.

Row Equivalent matrices: can get to the other matrix with row operations

$$A \sim B$$

To find row space use the nonzero rows of the RREF form of A.

Example 4.2.6. Let

$$A = \begin{bmatrix} -5 & -15 & 2 & 1 & -38 \\ -3 & -9 & 1 & 1 & -23 \\ -2 & -6 & 2 & -2 & -14 \end{bmatrix}.$$

use pivot columns IN ORIGINAL.

Find Nul A, Col, Row A and the simplest form of Col A. The reduced echelon form of A is

$$\begin{array}{ccccc} x_1 & x_2 & x_3 & x_4 & x_5 \end{array}$$
$$\left[\begin{array}{ccccc|c} 1 & 3 & 0 & -1 & 8 & 0 \\ 0 & 0 & 1 & -2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$x_1 + 3x_2 - x_4 + 8x_5 = 0$

$x_3 - 2x_4 + x_5 = 0$

$x_1 = -3x_2 + x_4 - 8x_5$

$x_2 = x_2$

$x_3 = 2x_4 - x_5$

$x_4 = x_4$

$x_5 = x_5$

$$\boxed{\operatorname{nul} A = \left\{ \underset{x_2}{\begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}}, \underset{x_4}{\begin{bmatrix} 1 \\ 0 \\ 2 \\ 1 \\ 0 \end{bmatrix}}, \underset{x_5}{\begin{bmatrix} -8 \\ 0 \\ -1 \\ 0 \\ 1 \end{bmatrix}} \right\}}$$

$x_4 = x_4$

$x_5 = x_5$

$x_2 \quad x_4 \quad x_5$

$$\text{Col } A = \left\{ \begin{bmatrix} -5 \\ -3 \\ -2 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix} \right\}$$

$$\text{Row } A = \{ \langle 1, 3, 0, -1, 8 \rangle, \langle 0, 0, 1, -2, 1 \rangle \}$$

Example 4.2.7. Find the row space, column space ~~and null space~~ of $A = \begin{bmatrix} 1 & 2 & 0 & -1 \\ 0 & -1 & 1 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}$

Need RREF for Nul A

$$\begin{bmatrix} 1 & 2 & 0 & -1 \\ 0 & -1 & 1 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix} \xrightarrow{R_1 - R_3 \to R_3} \begin{bmatrix} 1 & 2 & 0 & -1 \\ 0 & -1 & 1 & 1 \\ 0 & -1 & 1 & 1 \end{bmatrix} \xrightarrow{R_2 - R_3} \begin{bmatrix} 1 & 2 & 0 & -1 \\ 0 & -1 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\text{Col } A = \left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} \right\}$$

$$\text{Row } A = \{ [1 \ 2 \ 0 \ -1], [0 \ -1 \ 1 \ 1] \}$$