Section 4.1

Monday, October 3, 2022 1:44 PM

4.1 Vector Spaces and Subspaces

Vector Space

Definition 4.1. A non-empty set V is called a **vector space** if there are defined on V two operations, addition of vectors and multiplication by scalars, so that ten basic properties hold for all vectors in the space. These properties are listed here, (you are not required to memorize them),

Let u, v, and w be vectors in V, and let c and d be scalars (real numbers).

numbers

1. $u + v$ is in V.
2. cu is in V.

closure properties.

3. $u + v = v + u$.
4. $(u + v) + w = u + (v + w)$
5. $0 + u = u = u + 0$.
6. There exists a vector v such that $u + (v) = u + v = 0$. additive inverse
7. $c(u + v) = cu + cv$.
8. $(c + d)u = cu + du$.
9. $c(du) = (cd)u$.
10. $1u = u$.

That list of properties is long but here is a summary:

- The sum of any two vectors in V is also in V.
- Any scalar multiple of a vector in V is also in V. This includes $0v$.
- The last 8 can be stated as "Addition and scalar multiplication are well behaved."

$\mathbb{Z}$ integers.

$\mathbb{Z}^2$ is vector space?

No because $\frac{1}{2}\begin{pmatrix}1\\1\end{pmatrix} = \begin{pmatrix}1/2\\1/2\end{pmatrix} \notin \mathbb{Z}^2$

Example 4.1.1. Examples of vector spaces:

1. $\mathbb{R}^n$ $\mathbb{R}^3$ $u = \begin{pmatrix}a\\b\\c\end{pmatrix}$ $v = \begin{pmatrix}d\\e\\f\end{pmatrix}$ $a, b, c, d, e, f \in \mathbb{R}$
2. $\mathbb{P}_n$, the set of polynomials of degree at most n.

$u + v = \begin{pmatrix}a+d\\b+e\\c+f\end{pmatrix}$ $a+d,\ b+e,\ c+f \in \mathbb{R}$ ✓

$\mathbb{P}_4$ $v = x^4 + 5x^3 - 127.3542x$ $\mathbb{P}_4$ $\mathbb{R}^5$

$u = 7$

$w = 0$ zero vector ✓

$u + v$ degree 4 ✓

$$\{\hat{i}, \hat{j}, \hat{k}\}$$

$$\mathbb{R}^3 = \left\{ \begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\end{bmatrix} \right\}$$

Subspaces

Definition 4.2. A **subspace** H of a vector space V is a subset of V which is also a vector space. This means that H must contain the zero vector and must be closed under addition and scalar multiplication.

V is a subspace of V

Example 4.1.2. $H = \text{Span}\left\{ \begin{bmatrix}1\\0\\1\end{bmatrix}, \begin{bmatrix}0\\1\\1\end{bmatrix} \right\}$ is a subspace of $\mathbb{R}^3$ yes.

Check the properties: Given two vectors $u = a_1 \begin{bmatrix}1\\0\\1\end{bmatrix} + a_2 \begin{bmatrix}0\\1\\1\end{bmatrix}$ and $v = b_1 \begin{bmatrix}1\\0\\1\end{bmatrix} + b_2 \begin{bmatrix}0\\1\\1\end{bmatrix}$

Closed? Is $u + v$ in H?
Zero vector? ✓

$$u+v = a_1\begin{bmatrix}1\\0\\1\end{bmatrix} + a_2\begin{bmatrix}0\\1\\1\end{bmatrix} + b_1\begin{bmatrix}1\\0\\1\end{bmatrix} + b_2\begin{bmatrix}0\\1\\1\end{bmatrix}$$

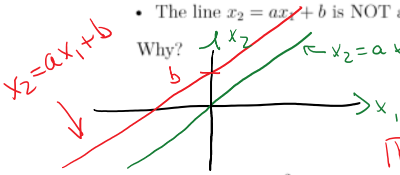
$$= (a_1+b_1)\begin{bmatrix}1\\0\\1\end{bmatrix} + (a_2+b_2)\begin{bmatrix}0\\1\\1\end{bmatrix} \checkmark$$

$$0\begin{bmatrix}1\\0\\1\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$$

Example 4.1.3. Some lines are subspaces and some are not.

- The line $x_2 = ax_1$ is a subspace of $\mathbb{R}^2$.
- The line $x_2 = ax_1 + b$ is NOT a subspace of $\mathbb{R}^2$. $b \neq 0$

Why?



$x_2 = ax_1$

$$X = \begin{pmatrix}x_1\\x_2\end{pmatrix} = \begin{pmatrix}1\\a\end{pmatrix} x_1$$

$x_1 = x_1$

$x_2 = ax_1$

Not subspace of $\mathbb{R}^2$

$x_2 = a x_1 + b$

$$X = \begin{pmatrix}x_1\\x_2\end{pmatrix} = \begin{pmatrix}0\\b\end{pmatrix} + \begin{pmatrix}1\\a\end{pmatrix} x_1$$

No value of x_1 will make $X = \begin{pmatrix}0\\0\end{pmatrix}$

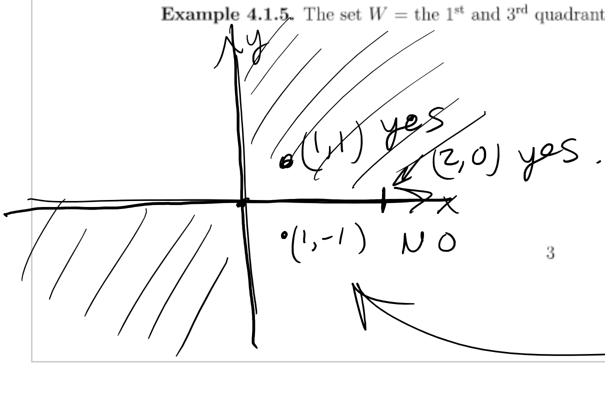
$\mathbb{R}^2$ is not a subset of $\mathbb{R}^3$

Example 4.1.4. Is $\mathbb{R}^2$ a subspace of $\mathbb{R}^3$?

$\begin{pmatrix}a\\b\end{pmatrix}$ No $\begin{pmatrix}a\\b\\c\end{pmatrix}$

Example 4.1.5. The set W = the 1[st] and 3[rd] quadrants of the plane. Is W a subspace of $\mathbb{R}^2$? NO

zero vector?



$(1,1) + (2,0) = (3, 1)$ ✓ In the space

$(2,0) + (-1,-1) = (1,-1)$

NOT closed under addition.

Basis

Definition 4.3. A **basis** for a vector space is a set of linearly independent vectors that generate the space.

A basis is a minimal spanning set.

Example 4.1.6.

$\left\{ \begin{bmatrix}1\\1\end{bmatrix}, \begin{bmatrix}2\\1\end{bmatrix}, \begin{bmatrix}1\\2\end{bmatrix} \right\}$ is NOT a basis for $\mathbb{R}^2$

Not linearly independent

$\left\{ \begin{bmatrix}1\\1\end{bmatrix}, \begin{bmatrix}2\\1\end{bmatrix} \right\}$ is a basis for $\mathbb{R}^2$

$$\begin{bmatrix}1 & 2\\1 & 1\end{bmatrix} \xrightarrow{RR} \begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}$$

Example 4.1.7. Is the set of vectors of the form $\begin{bmatrix}a-b\\b-c\\c-a\\b\end{bmatrix}$ a vector space? If it is, find a basis.

closed

zero vector.

4 entries

$$\begin{bmatrix}a-b\\b-c\\c-a\\b\end{bmatrix} = \begin{bmatrix}1\\0\\-1\\0\end{bmatrix}a + \begin{bmatrix}-1\\1\\0\\1\end{bmatrix}b + \begin{bmatrix}0\\-1\\1\\0\end{bmatrix}c$$

Also subspace of $\mathbb{R}^4$

$$\text{Basis} = \left\{ \begin{bmatrix}1\\0\\-1\\0\end{bmatrix}, \begin{bmatrix}-1\\1\\0\\1\end{bmatrix}, \begin{bmatrix}0\\-1\\1\\0\end{bmatrix} \right\}$$

Example 4.1.8. Is the set of vectors of the form $\begin{bmatrix}3a+b\\4\\a-5b\end{bmatrix}$ a vector space? If it is, find a basis.

$$\begin{bmatrix}3a+b\\4\\a-5b\end{bmatrix} = \begin{bmatrix}3\\0\\1\end{bmatrix}a + \begin{bmatrix}1\\0\\-5\end{bmatrix}b + \begin{bmatrix}0\\4\\0\end{bmatrix}$$

Not vector space b/c no zero vector.