Section 3.3

Wednesday, September 28, 2022 1:13 PM

3.3 Cramer's Rule, Volume, and Linear Transformations

3.3.1 Cramer's Rule

Notation: For any $x \times n$ matrix A and any b in $\mathbb{R}^n$, let $A_i(b)$ be the matrix obtained from A by replacing column i by the vector b.

$$A_i(b) = [a_1 \ \cdots \ a_{i-1} \ \ b \ \ a_{i+1} \ \cdots \ a_n] \leftarrow n \text{ columns}$$

$\uparrow$ i^{th} column replaced with b

Example 3.3.1. $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$ then

replace

$$A_1\begin{pmatrix} -5 \\ -6 \\ -7 \end{pmatrix} = \begin{bmatrix} -5 & 2 & 3 \\ -6 & 5 & 6 \\ -7 & 8 & 9 \end{bmatrix}$$

Cramer's Rule

Theorem 3.5. Let A be an invertible $x \times n$ matrix. For any b in $\mathbb{R}^n$, the unique solution x of $Ax = b$ has entries given by

$$x_i = \frac{\det A_i(b)}{\det A}$$

Example 3.3.2. Use Cramer's Rule to solve the system of equations

$$\begin{aligned} 4x_1 + x_2 &= 6 \\ 3x_1 + 2x_2 &= 7 \end{aligned} \leftarrow b$$

$A = \begin{bmatrix} 4 & 1 \\ 3 & 2 \end{bmatrix}$ $\qquad$ $b = \begin{bmatrix} 6 \\ 7 \end{bmatrix}$

$A_1(b) = \begin{bmatrix} 6 & 1 \\ 7 & 2 \end{bmatrix}$ $\qquad$ $A_2(b) = \begin{bmatrix} 4 & 6 \\ 3 & 7 \end{bmatrix}$

$\det A = 8 - 3 = 5$ $\qquad$ $\det A_1(b) = 12 - 7 = 5$ $\qquad$ $\det A_2(b) = 28 - 18 = 10$

$$x_1 = \frac{\det(A_1(b))}{\det A} = \frac{5}{5} = 1$$

$$x_2 = \frac{\det(A_2(b))}{\det A} = \frac{10}{5} = 2$$

Example 3.3.3. Use Cramer's Rule to determine the values of the parameter s for which the system has a unique solution and describe the solution.

$$s x_1 - 6 s x_2 = 3$$
$$3 x_1 - 18 s x_2 = 5$$

b

$$x_1 = \frac{\det A_1(b)}{\det A} = \frac{4}{3(s-1)} = \frac{-24s}{-18s(s-1)}$$

$$\boxed{x_2 = \frac{5s-9}{-18s(s-1)}}$$

$s \neq 0 \quad s \neq 1$

$$A = \begin{bmatrix} s & -6s \\ 3 & -18s \end{bmatrix} \qquad A_1(b) = \begin{bmatrix} 3 & -6s \\ 5 & -18s \end{bmatrix}$$

$$\det A = -18s^2 + 18s = -18s(s-1)$$

$$\det A_1(b) = -54s + 30s = -24s$$

$$A_2(b) = \begin{bmatrix} s & 3 \\ 3 & 5 \end{bmatrix} \qquad \det A_2(b) = 5s - 9$$

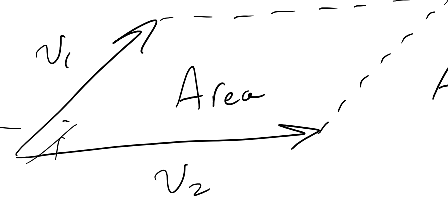
3.3.2 Area and Volume

Area and Volume

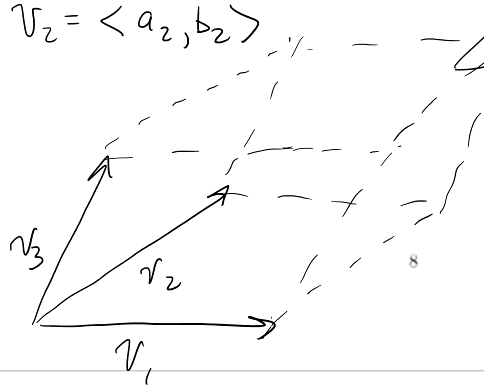
Theorem 3.6.
Area: If A is a 2×2 matrix, the area of the parallelogram determined by the columns of A is $|\det A|$.

Volume: If A is a 3 matrix, the volume of the parallelepiped determined by the columns of A is $|\det A|$.

$v_1 = \langle a_1, b_1 \rangle$
$v_2 = \langle a_2, b_2 \rangle$



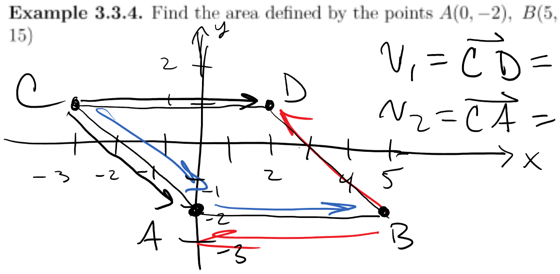
$$A = \det \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}$$



$$V = \begin{vmatrix} v_1 & v_2 & v_3 \end{vmatrix}$$

$$\det A = \det A^T$$

Example 3.3.4. Find the area defined by the points $A(0,-2)$, $B(5,-2)$, $C(-3,1)$, $D(2,1)$. (ans 15)



$$v_1 = \vec{CD} = D - C = (2,1) - (-3,1) = \langle 5, 0 \rangle$$
$$v_2 = \vec{CA} = A - C = (0,-2) - (-3,1) = \langle 3, -3 \rangle$$

$$\det = |5(-3) - 0| = |-15| = 15$$

Example 3.3.5. Find the volume of the parallelepiped with one vertex at the origin and adjacent vertices at $(1,0,-3)$, $(1,4,4)$, and $(8,2,0)$ (ans. 82)

$(1,0,-3)$

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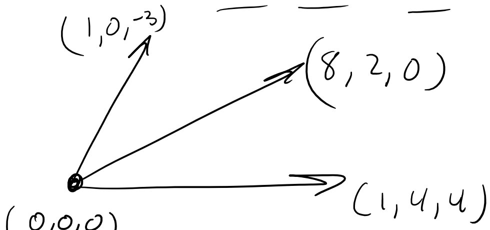

$$V = \begin{vmatrix} 1 & 1 & 8 \\ 0 & 4 & 2 \\ -3 & 4 & 0 \end{vmatrix} = (1)\begin{vmatrix} 4 & 2 \\ 4 & 0 \end{vmatrix} - 0 + (-3)\begin{vmatrix} 1 & 8 \\ 4 & 2 \end{vmatrix}$$

$$= -8 - 3(2-32) = -8+90$$

$$\boxed{= 82}$$

3.3.3 Linear Transformations

Linear Transformations with Area and Volume

Theorem 3.7. Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation determined by a ~~3~~ 2 $\times$ 2 matrix A. If S is the parallelogram in $\mathbb{R}^2$, then

det of transformation

$$\{\text{area of } T(S)\} = |\det A| \cdot \{\text{area of } S\}$$

Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be the linear transformation determined by a 3×3 matrix A. If S is the parallelepiped in $\mathbb{R}^3$, then

$$\{\text{volume of } T(S)\} = |\det A| \cdot \{\text{volume of } S\}$$

$$T(x) = \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}$$

$$T\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \end{bmatrix}$$

Example 3.3.6. Let S be the parallelogram determined by the vectors $b_1 = \begin{bmatrix} -3 \\ 6 \end{bmatrix}$ and $b_2 = \begin{bmatrix} -3 \\ 10 \end{bmatrix}$, and let $A = \begin{bmatrix} 6 & -3 \\ -5 & 3 \end{bmatrix}$. Compute the area of the image of S under the mapping $x \mapsto Ax$. (ans. 36)

$$\text{Area of } S = \begin{vmatrix} -3 & -3 \\ 6 & 10 \end{vmatrix} = |-30+18| = +12$$

$$\text{Area of } T(S) = 3(12) = \boxed{36}$$

$$\det A = \begin{vmatrix} 6 & -3 \\ -5 & 3 \end{vmatrix} = 18 - 15 = 3$$