Section 3.2

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3.2 Properties of Determinants

Row Operations and Determinants

Theorem 3.2. Let A be a square matrix.

1. If two rows of A are interchanged to produce B, then $\det B = -\det A$.
2. If one row of A is multiplied by k to produce B, then $\det B = k \det A$.
3. If a multiple of one row of A is **added** to another row to produce matrix B, then $\det B = \det A$.

$2R_2+R_3$ Must replace row 3

R_2+3R_3 Must replace row 2

$2R_1+3R_3$ Don't do it.

Example 3.2.1. Rule # 1: If you switch two rows the sign of the determinant changes.

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \qquad B = \begin{bmatrix} 3 & 4 \\ 1 & 2 \end{bmatrix}$$

$\det A = 4(1) - 3(2) = -2$ $\quad \det B = 6 - 4 = +2$

Example 3.2.2. Rule # 2: If you mulitply a row of matrix A by a number, c, to make matrix B then $\det B = c \det A$.

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 3 & 1 \\ 0 & 0 & -3 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1/3 \\ 0 & 0 & -3 \end{bmatrix}$$

$B = \frac{1}{3}R_2$ of A

$\det A = -9$ $\quad \det B = \frac{1}{3}\det A = -3$

$3 \det B = \det A$

Example 3.2.3. Rule # 3: If a multiple of one row of A is **added** to another row to produce matrix B the determinants are the same: $\det B = \det A$.

$$c * R_i + R_j \longrightarrow R_j$$

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & -3 & 1 \\ 2 & 2 & -1 \end{bmatrix} \xrightarrow{-2R_1+R_3} B = \begin{bmatrix} 1 & 1 & 1 \\ 0 & -3 & 1 \\ 0 & 0 & -3 \end{bmatrix}$$

$\det A = \det B = 9$ ✓

$\det A = -0\begin{vmatrix} 1 & 1 \\ 2 & -1 \end{vmatrix} + (-3)\begin{vmatrix} 1 & 1 \\ 2 & -1 \end{vmatrix} - 1\begin{vmatrix} 1 & 1 \\ 2 & 2 \end{vmatrix}$

$= -3(-1-2) - 1(2-2) = 9$ ✓

Other properties of determinants

Theorem 3.3. Let A be a square matrix. $(n \times n)$ $\quad k$ is a constant.

1. $\det(A^{-1}) = \dfrac{1}{\det A}$.
2. $\det(AB) = (\det A)(\det B)$
3. $\det(kA) = \cancel{k(\det A)}$ $k^n(\det A)$

det A constant

2. det $(AB) = (\det A)(\det B)$
3. det $(kA) = \cancel{k(\det A)}$ $k^n(\det A)$
4. det $A^{\mathrm{T}} = \det A$

Invertible Matrices and Determinants

Theorem 3.4. A square matrix A is invertible if and only if det $A \neq 0$.

Example 3.2.4. Find det A

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 5 & 6 & 7 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 5 & 6 & 7 \end{bmatrix} \xrightarrow[-5R_1+R_3]{-2R_1+R_2} \begin{bmatrix} 1 & 2 & 3 \\ 0 & -1 & -2 \\ 0 & -4 & -8 \end{bmatrix} \xrightarrow{-4R_2+R_3} \begin{bmatrix} 1 & 2 & 3 \\ 0 & -1 & -2 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\det A = 0$$

[ ] matrix

| | determinant

Example 3.2.5. Find det A

$$A = \begin{bmatrix} 2 & 0 & 0 & 6 \\ 1 & -7 & -5 & 0 \\ 3 & 8 & 6 & 0 \\ 0 & 7 & 5 & 4 \end{bmatrix}.$$

$R_2 - R_1$
$R_3 - 3R_1$

$$\det A = 2\begin{vmatrix} 1 & 0 & 0 & 3 \\ 0 & -7 & -5 & -3 \\ 0 & 8 & 6 & -9 \\ 0 & 7 & 5 & 4 \end{vmatrix} = 2(1)\begin{vmatrix} -7 & -5 & -3 \\ 8 & 6 & -9 \\ 7 & 5 & 4 \end{vmatrix} = 2(1)\begin{array}{c} \begin{array}{ccc} + & - & + \end{array} \\ \begin{vmatrix} 0 & 0 & 1 \\ 8 & 6 & -9 \\ 7 & 5 & 4 \end{vmatrix} \end{array}$$

$R_3 + R_1$

$$= 2(1)(1)\begin{vmatrix} 8 & 6 \\ 7 & 5 \end{vmatrix} = 2(40 - 42) = -4$$