Chapter3_Lay_handout

Monday, September 26, 2022 1:44 PM

3.1 Introduction to Determinants

2 x 2 Matrix Determinant

Definition 3.1. Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and define the determinant of A as $\det A = ad - bc$.

Definition 3.2. For any square matrix A, let A_{ij} denote the submatrix formed by deleting the i^{th} row and j^{th} column of A

2nd column

Example 3.1.1. $A = \begin{bmatrix} 3 & 4 & -5 & -2 \\ 2 & -3 & 5 & -1 \\ 3 & 0 & 5 & 0 \\ 4 & 9 & 4 & 5 \end{bmatrix}$ Find A_{32}, A_{23} and A_{44}

3rd row

$a_{32} = 0$

$A_{32} = \begin{bmatrix} 3 & -5 & -2 \\ 2 & 5 & -1 \\ 4 & 4 & 5 \end{bmatrix}$ $A_{23} = \begin{bmatrix} 3 & 4 & -2 \\ 3 & 0 & 0 \\ 4 & 9 & 5 \end{bmatrix}$ $A_{44} = \begin{bmatrix} 3 & 4 & -5 \\ 2 & -3 & 5 \\ 3 & 0 & 5 \end{bmatrix}$

1st row entry

sub matrix

[ ] matrix

| | det matrix

The Determinant

Definition 3.3. For $n \geq 2$ the **determinant** of an $x \times n$ matrix $A = [a_{ij}]$ is the sum of n terms of the form $\pm a_{1j} \boxed{\det A_{1j}}$, with plus and minus signs alternating, where the entries $a_{11}, a_{12}, \ldots, a_{1n}$ are from the first row of A. In symbols,

$$\begin{aligned} \det A &= a_{11}\det A_{11} - a_{12}\det A_{12} + a_{13}\det A_{13} - \cdots + (-1)^{1+n} a_{1n}\det A_{1n} \\ &= \sum_{j=1}^{n} (-1)^{1+j} a_{1j} \det A_{1n} \end{aligned}$$

Example 3.1.2. For a 3×3 matrix $A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$ that would look like: 1st row expansion

$$\det A = a\begin{vmatrix} e & f \\ h & i \end{vmatrix} - b\begin{vmatrix} d & f \\ g & i \end{vmatrix} + c\begin{vmatrix} d & e \\ g & h \end{vmatrix} = a(ei - hf) - b(di - fg) + c(dh - eg)$$

$a_{11} \det A_{11} - a_{12} \det A_{12} + a_{13} \det A_{13}$

$\det \begin{vmatrix} d & e \\ g & h \end{vmatrix}$

Example 3.1.3. Find the determinant of $A = \begin{bmatrix} 3 & 4 & -5 \\ 2 & -3 & 5 \\ 3 & 0 & 5 \end{bmatrix}$

$$\det A = 3\begin{vmatrix} -3 & 5 \\ 0 & 5 \end{vmatrix} - 4\begin{vmatrix} 2 & 5 \\ 3 & 5 \end{vmatrix} - 5\begin{vmatrix} 2 & -3 \\ 3 & 0 \end{vmatrix}$$

$$= 3(-15-0) - 4(10-15) - 5(0-(-9))$$

$$= -45 + 20 - 45 = -70$$

$$\left[\begin{matrix} 3 & 4 & -5 \\ 2 & -3 & 5 \\ 3 & 0 & 5 \end{matrix}\right] \begin{matrix} 3 & 4 \\ 2 & -3 \\ 3 & 0 \end{matrix}$$

$$- \quad - \quad - \quad + \quad + \quad +$$

$$-45 + 60 - 40 - 45 = -70$$

Example 3.1.4. Find the determinant of $A = \begin{bmatrix} 3 & 4 & -5 & 0 \\ 2 & -3 & 5 & -1 \\ 3 & 0 & 5 & 0 \\ 4 & 9 & 0 & 5 \end{bmatrix}$

$$\det A = 3\begin{vmatrix} -3 & 5 & -1 \\ 0 & 5 & 0 \\ 9 & 0 & 5 \end{vmatrix} - 4\begin{vmatrix} 2 & 5 & -1 \\ 3 & 5 & 0 \\ 4 & 0 & 5 \end{vmatrix} + (-5)\begin{vmatrix} 2 & -3 & -1 \\ 3 & 0 & 0 \\ 4 & 9 & 5 \end{vmatrix} + 0\begin{vmatrix} & & \end{vmatrix}$$

Not going to finish

start –

Example 3.1.5. Find the determinant of $A = \begin{bmatrix} 4 & 0 & -7 & 3 & -5 \\ 0 & 0 & 2 & 0 & 0 \\ 7 & 3 & -6 & 4 & -8 \\ 5 & 0 & 5 & 2 & -3 \\ 0 & 0 & 9 & -1 & 2 \end{bmatrix}$ ← use row 2

$$\det A = -2 \begin{vmatrix} 4 & 0 & 3 & -5 \\ 7 & 3 & 4 & -8 \\ 5 & 0 & 2 & -3 \\ 0 & 0 & -1 & 2 \end{vmatrix} = \underbrace{(-2)(+3)}_{(-6)} \begin{vmatrix} 4 & 3 & -5 \\ 5 & 2 & -3 \\ 0 & -1 & 2 \end{vmatrix} = (-6)\left[4 \begin{vmatrix} 2 & -3 \\ -1 & 2 \end{vmatrix} - 5 \begin{vmatrix} 3 & -5 \\ -1 & 2 \end{vmatrix}\right]$$

$$= (-6)\left[4(4-3) - 5(6-5)\right] = \boxed{6}$$

Example 3.1.6. Find the determinant of the upper triangular matrix

$$A = \begin{bmatrix} a_1 & x & x & x & x \\ 0 & a_2 & x & x & x \\ 0 & 0 & \ddots & x & x \\ \vdots & \vdots & \ddots & \ddots & x \\ 0 & 0 & \cdots & 0 & a_n \end{bmatrix} = a_1 \begin{vmatrix} a_2 & x & x \\ 0 & \ddots & \\ 0 & & a_n \end{vmatrix} - x_1 \begin{vmatrix} 0 & x & x & x \\ 0 & & & x \\ 0 & & x & \\ 0 & & x & x \\ 0 & & & \end{vmatrix}$$

← det = 0

$$\det A = \underbrace{a_1 a_2 \cdots a_n}_{\text{diagonal elements}}$$

Cofactor Expansions

Theorem 3.1.

Given $n \times n$ matrix $A = [a_{ij}] = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$

Then the matrix A_{ij} = is the matrix formed by deleting the i^{th} row and j^{th} column of A
The (i, j)**-cofactor** of A is the number C_{ij} given by

$$C_{ij} = (-1)^{i+j} \det A_{ij}$$

then

$$\det A = a_{i1}C_{i1} + a_{i2}C_{i2} + \cdots + a_{in}C_{in}$$

This formula is called **the cofactor expansion across the i^{th} row** of A.

A similar formula can be constructed for **the cofactor expansion down the j^{th} column** of A:

$$\det A = a_{1j}C_{1j} + a_{2j}C_{2j} + \cdots + a_{nj}C_{nj}$$