Section2.3_Lay6e

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2.3 Characterizations of Invertible Matrices

Characterizations of invertible matrices

Theorem 2.6. Let A be a square $n \times n$ matrix. Then the following statements are equivalent. That is, for a given A, the statements are either all true or all false.

a. A is an invertible matrix.

b. A is row equivalent to I_n

c. A has n pivot positions.

d. The equation $Ax = 0$ has only the trivial solution. ← From our knowledge about homogeneous equations.

e. The columns of A form a linearly independent set.

f. The linear transformation $x \mapsto Ax$ is one-to-one.

g. The equation $Ax = b$ has at least one solution for each b in $\mathbb{R}^n$.

h. The columns of A span $\mathbb{R}^n$.

i. The linear transformation $x \mapsto Ax$ maps $\mathbb{R}^n$ onto $\mathbb{R}^n$.

j. There is an $n \times n$ matrix C such that $CA = I$. $C = A^{-1}$

k. There is an $n \times n$ matrix D such that $AD = I$. $D = A^{-1}$

l. A^T is an invertible matrix.

$Ax = b$ solution

$$x = \underbrace{x_p}_{\text{unique}} + \overbrace{x_n}^{\text{homogeneous solution}}$$

Infinite

$(\ )t + (\ )s + \ldots$

Example 2.3.1. All matrices in this example are $n \times n$. Each part of the example is an implication of the form If "statement 1" then "statement 2". The implication is TRUE if "statement 2" is ALWAYS true whenever "statement 1" happens. If there is a time when it is not true then it is FALSE.

1. If $\underbrace{\text{the equation } Ax = 0 \text{ has only the trivial solution}}_{d}$, then $\underbrace{A \text{ is row equivalent to } I_n}_{b}$. True

2. If $\underbrace{\text{the columns of } A \text{ span } \mathbb{R}^n}_{h}$, then $\underbrace{\text{the columns are linearly independent}}_{e}$. True

3. If $\underbrace{\text{the equation } Ax = 0 \text{ has a nontrivial solution}}_{\text{Not } d}$, then A has fewer than n pivot positions. True

4. If A^T is not invertible, then A is not invertible. T

5. If there is an $n \times n$ matrix D such that $AD = I$, then there is also and $n \times n$ matrix C such that $CA = I$.

True

6. If the columns of A are linearly independent, then the columns of A span $\mathbb{R}^n$. True

e h

7. If the equation $Ax = b$ has at least one solution for each $b \in \mathbb{R}^n$, then the solution is unique for each b. True

Since there is a solution for each b in $\mathbb{R}^n$ means that A spans $\mathbb{R}^n$

Invertible Transformations

Theorem 2.7. Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be a linear transformation and let A be the standard matrix for T. Then T is invertible if and only if A is an invertible matrix. In that case, the linear transformation S given by $S(x) = A^{-1}x$ is the unique function satisfying $T(S(x)) = x$ and $S(T(x)) = x$.

$T(x) = Ax$

$T^{-1}(x) = A^{-1}x$

Example 2.3.2. T is a linear transformation from $\mathbb{R}^2$ into $\mathbb{R}^2$. Show that T is invertible and find a formula for T^{-1}.

$$T(x_1, x_2) = (-5x_1 + 9x_2, 4x_1 - 7x_2) = \begin{pmatrix} -5x_1 + 9x_2 \\ 4x_1 - 7x_2 \end{pmatrix}$$

Find A^{-1}

$$\left[\begin{array}{cc|cc} -5 & 9 & 1 & 0 \\ 4 & -7 & 0 & 1 \end{array}\right] \xrightarrow[-R_1]{R_1+R_2 \to R_1} \left[\begin{array}{cc|cc} +1 & -2 & -1 & -1 \\ 4 & -7 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{R_2-4R_1 \to R_2} \left[\begin{array}{cc|cc} 1 & -2 & -1 & -1 \\ 0 & 1 & 4 & 5 \end{array}\right] \xrightarrow{R_1+2R_2} \left[\begin{array}{cc|cc} 1 & 0 & 7 & 9 \\ 0 & 1 & 4 & 5 \end{array}\right]$$

A^{-1}

...t $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and define the determinant ...vertible and

$$A^{-1} = \frac{1}{\det A}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

...atrix is not invertible.

$\det A = (-5)(-7) - (4)(9) = -1$

$$\frac{1}{-1}\begin{bmatrix} -7 & -9 \\ -4 & -5 \end{bmatrix} = \begin{bmatrix} 7 & 9 \\ 4 & 5 \end{bmatrix}$$