Section2.2_Lay6e

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2.2 The Inverse of a Matrix

Recall: The **Identity Matrix** is a square matrix with 1's along the main diagonal and zeros everywhere else.

Example 2.2.1. $I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $I_4 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

$AI = IA = A$

We call it the identity matrix because it behaves like 1 in multiplication.

What is the inverse of 7?

Example 2.2.2. $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ $\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

$7 - 7 = 0$ ← additive identity

Example 2.2.3. $\begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix}\underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{I_3} = \begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix}$

$\underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}_{I_2}\begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix}$

2.2.1 Matrix Inverses

$7\left(\frac{1}{7}\right) = 1$ ← multiplicative identity

Definition 2.3. If M is a square matrix and if there exists M^{-1} such that

$$MM^{-1} = I \text{ and } M^{-1}M = I$$

then M^{-1} is the **Multiplicative Inverse** of M. We often simply call it "The Inverse" of M.

Example 2.2.4. $A = \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix}$ and $A^{-1} = \begin{bmatrix} 2 & -3 \\ -1 & 2 \end{bmatrix}$. Show that these are inverses of each other.

$$\begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 2 & -3 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 4-3 & -6+6 \\ 2-2 & -3+4 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$A = \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix}$ $\det A = 4 - 3 = 1$

NOT ALL SQUARE MATRICES HAVE INVERSES. For example $\begin{bmatrix} 2 & 1 \\ 4 & 2 \end{bmatrix}$

$A^{-1} = \frac{1}{\det A}\begin{bmatrix} 2 & -3 \\ -1 & 2 \end{bmatrix}$

Q. How do we know if an inverse exists for $[A]$ and how do we find one if it does?

A. We perform Gauss Jordan Elimination on the augmented matrix $\left[\, A \mid I \,\right]$ until it looks like

$$\left[\, I \mid A^{-1} \,\right]$$

If a matrix does NOT have an inverse we call it a **singular matrix**.

$$\begin{bmatrix} 1 & 0 \\ -3 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Example 2.2.5. $A = \begin{bmatrix} 1 & 0 \\ -3 & 1 \end{bmatrix}$ Find A^{-1}

Row reduce this matrix:

$$[A|I] = \left[\begin{array}{cc|cc} 1 & 0 & 1 & 0 \\ -3 & 1 & 0 & 1 \end{array}\right] \xrightarrow{3R_1+R_2 \to R_2} \left[\begin{array}{cc|cc} 1 & 0 & 1 & 0 \\ 0 & 1 & 3 & 1 \end{array}\right]$$

Row Reduce to $[I|A^{-1}]$

$$A^{-1} = \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix}$$

2 x 2 Matrix Inverse

Theorem 2.3. Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and define the determinant of A as $\det A = ad - bc$. If det $A \neq 0$ then A is invertible and

det

$$A^{-1} = \frac{1}{\det A}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

If det $A = 0$ the matrix is not invertible.

Example 2.2.6. $A = \begin{bmatrix} 3 & 9 \\ 2 & 6 \end{bmatrix}$ Find A^{-1}

$\det A = 18 - 18 = 0$ Done A^{-1} DNE

Start w/ 1 in 1st position

Example 2.2.7. $A = \begin{bmatrix} 1 & 5 & 10 \\ 0 & 1 & 4 \\ 1 & 6 & 15 \end{bmatrix}$ Find A^{-1}

Row reduce this matrix:

zeros here

zeros

$$\left[\begin{array}{ccc|ccc} 1 & 5 & 10 & 1 & 0 & 0 \\ 0 & 1 & 4 & 0 & 1 & 0 \\ 1 & 6 & 15 & 0 & 0 & 1 \end{array}\right] \xrightarrow{R_3 - R_1 \to R_3} \left[\begin{array}{ccc|ccc} 1 & 5 & 10 & 1 & 0 & 0 \\ 0 & 1 & 4 & 0 & 1 & 0 \\ 0 & 1 & 5 & -1 & 0 & 1 \end{array}\right] \xrightarrow[R_3 - R_2 \to R_3]{R_1 - 5R_2 \to R_1} \left[\begin{array}{ccc|ccc} 1 & 0 & -10 & 1 & -5 & 0 \\ 0 & 1 & 4 & 0 & 1 & 0 \\ 0 & 0 & 1 & -1 & -1 & 1 \end{array}\right]$$

zeros

$$\xrightarrow[R_1 + 10R_3 \to R_1]{R_2 - 4R_3 \to R_2} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -9 & -15 & 10 \\ 0 & 1 & 0 & 4 & 5 & -4 \\ 0 & 0 & 1 & -1 & -1 & 1 \end{array}\right]$$

A^{-1}

Example 2.2.8. $M = \begin{bmatrix} 3 & -1 & 1 \\ -1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$ Find $M^{-1} = \begin{bmatrix} 1 & 1 & -1 \\ 1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix}$

Solving a matrix equation

Suppose we have a system of equations

$$\begin{aligned} a_1x_1 + a_2x_2 + a_3x_3 &= a_4 \\ b_1x_1 + b_2x_2 + b_3x_3 &= b_4 \\ c_1x_1 + c_2x_2 + c_3x_3 &= c_4 \end{aligned}$$

where a_i, b_i, and c_i are real numbers and x_1, x_2, x_3 are variables. Then we can write the coefficient matrix

$$A = \begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix}$$

and (IF it exists) we can find the inverse matrix A^{-1}.

The original system can be written in matrix form:

$$\underbrace{\begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}}_{X} = \underbrace{\begin{bmatrix} a_4 \\ b_4 \\ c_4 \end{bmatrix}}_{b}$$

$a_1x_1 + a_2x_2 + a_3x_3 = a_4$

and we end up with an equation of the form $AX = b$. If this were an algebraic equation where A and b were numbers we could easily solve this by dividing on both sides by A. WE CAN'T divide matrices.

What we can do with matrices is to multiply by the inverse of A. Then we get something that looks like this

$A^{-1}A = I$

$$\begin{aligned} AX &= b \\ A^{-1}[AX] &= A^{-1}b \\ IX &= A^{-1}b \\ X &= A^{-1}b \end{aligned}$$

$\frac{1}{7}(7x) = 42\left(\frac{1}{7}\right)$

$x = 6$

The nice thing about solving an equation this way is that now we can easily solve many problems that have the same A but different b with one simple matrix multiplication.

Example 2.2.9. Solve

$$\begin{aligned} 3x_1 - x_2 + x_3 &= 1 \\ -x_1 + x_2 &= 1 \\ x_1 + x_3 &= 1 \end{aligned}$$

$0x_3$

$0x_2$

by writing the equation in matrix form as $AX = b$ and multiplying by A^{-1}.

$$A^{-1} = \begin{bmatrix} 1 & 1 & -1 \\ 1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix}$$

$$\begin{bmatrix} 3 & -1 & 1 \\ -1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$Ax = b$

$$\begin{bmatrix} 1 & 1 & -1 \\ 1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix} \left(\begin{bmatrix} 3 & -1 & 1 \\ -1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \right) = \begin{bmatrix} 1 & 1 & -1 \\ 1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$A^{-1}Ax = A^{-1}b$

$$\begin{bmatrix} 1 & 1 & -1 \\ 1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix}\begin{bmatrix} 3 & -1 & 1 \\ -1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \cdots \\ 1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix}\begin{bmatrix} \cdots \\ 1 \\ 1 \end{bmatrix} \qquad A^{-1}Ax = A^{-1}b$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1+1-1 \\ 1+2-1 \\ -1-1+2 \end{bmatrix} = \boxed{\begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} = x}$$

Example 2.2.10. Solve
$$\begin{aligned} 3x_1 - x_2 + x_3 &= 1 \\ -x_1 + x_2 &= 2 \\ x_1 + x_3 &= -3 \end{aligned}$$

$$x = A^{-1}b = \begin{bmatrix} 1 & 1 & -1 \\ 1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix} = \boxed{\begin{bmatrix} 6 \\ 8 \\ -9 \end{bmatrix} = x}$$

Properties of Invertible Matrices

Theorem 2.4. Properties of Invertible Matrices

1. If A is an invertible matrix, then A^{-1} is invertible and
$$(A^{-1})^{-1} = A$$
2. If A and B are $n \times n$ invetrible matrices, then so is AB, and the inverse of AB is the product of the inverses of A and B in reverse order:
$$(AB)^{-1} = B^{-1}A^{-1}$$
$$(AB)^T = B^T A^T$$
3. If A is an invertible matrix, then so is A^T, and the inverse of A^T is the transpose of A^{-1}:
$$(A^T)^{-1} = (A^{-1})^T$$

2.2.2 Elementary Matrices

Definition 2.4. An **elementary matrix** is one that is obtained by performing a single elementary row operation on an identity matrix

switch rows
multiply row by number
Linear combination of rows,

Example 2.2.11. Find the product E_1A and E_2A and identify the corresponding row operation where

$$E_1 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \quad E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 4 \\ 0 & 0 & 1 \end{bmatrix}, \quad \text{and} \quad A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$$

$$E_1A = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} = \begin{bmatrix} g & h & i \\ d & e & f \\ a & b & c \end{bmatrix}$$

1 row operation
switch two rows.

Invertible matrices are row equivalent to I_n

Theorem 2.5. An $n \times n$ matrix A is invertible if and only if A is row equivalent to I_n, and in this case, any sequence of elementary row operations that reduces A to I_n also transforms I_n into A^{-1}

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 4 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} = \boxed{\begin{bmatrix} a & b & c \\ d+4g & e+4h & f+4i \\ g & h & i \end{bmatrix}}$$

1 Row operation
$R_2 + 4R_3 \to R_2$