Section2.1_Lay6e

Wednesday, September 14, 2022 1:31 PM

2.1 Operations with Matrices

- Addition and Subtraction
- Multiplication by a scalar
- Multiplication by another matrix

2.1.1 Addition and Subtraction

Q. What does it mean for two matrices to be **equal**?

A. It means they are the same size and have the EXACT same entries.

We can only add and subtract matrices that are the same size.

Q. How do we add matrices?

A. We add corresponding entries.

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} + \begin{bmatrix} e & f \\ g & h \end{bmatrix} = \begin{bmatrix} (a+e) & (b+f) \\ (c+g) & (d+h) \end{bmatrix}$$

Example 2.1.1. $\begin{bmatrix} 3 & 2 \\ -1 & -1 \\ 0 & 3 \end{bmatrix} + \begin{bmatrix} -2 & 3 \\ 1 & -1 \\ 2 & -2 \end{bmatrix} = \begin{bmatrix} 1 & 5 \\ 0 & -2 \\ 2 & 1 \end{bmatrix}$

What about standard addition properties? Matrix addition is: well behaved.

1. Commutative: $A + B = B + A$.
2. Associative: $(A + B) + C = A + (B + C)$

Definition 2.1. The **Zero Matrix** is a matrix with all entries zero. We often use 0 to represent it.

Example 2.1.2.

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} + 0 = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & 3 & 4 \end{bmatrix} + 0 = \begin{bmatrix} 1 & 2 & 3 & 4 \end{bmatrix}$$

2.1.2 Multiplication by a scalar

Multiply every entry in the matrix by the number.

Example 2.1.3.

$$5\begin{bmatrix} -7 & 3 & 0 & 9 \\ 4 & -5 & 6 & 2 \end{bmatrix} = \begin{bmatrix} 5(-7) & 5(3) & 5(0) & 5(9) \\ 5(4) & 5(-5) & 5(6) & 5(2) \end{bmatrix}$$

$$= \begin{bmatrix} -35 & 15 & 0 & 45 \\ 20 & -25 & 30 & 10 \end{bmatrix}$$

Example 2.1.4. $A = \begin{bmatrix} 3 & 2 & 0 \\ -1 & 4 & -6 \end{bmatrix}$ $\quad B = \begin{bmatrix} 5 & -2 & 7 \\ 1 & 3 & -9 \end{bmatrix}$

Find $-2A - B$ and $4B - A$

$$-2A - B = -2\begin{bmatrix} 3 & 2 & 0 \\ -1 & 4 & -6 \end{bmatrix} - \begin{bmatrix} 5 & -2 & 7 \\ 1 & 3 & -9 \end{bmatrix}$$

$$= \begin{bmatrix} -11 & -2 & -7 \\ 1 & -11 & 21 \end{bmatrix}$$

$$4B - A = \begin{bmatrix} 20 & -8 & 28 \\ 4 & 12 & -36 \end{bmatrix} - \begin{bmatrix} 3 & 2 & 0 \\ -1 & 4 & -6 \end{bmatrix} = \begin{bmatrix} 17 & -10 & 28 \\ 5 & 8 & -30 \end{bmatrix}$$

2.1.3 Multiplication of two matrices

An $n \times 1$ matrix multiplied by a $1 \times n$ matrix is the 1×1 matrix given by:

$$\begin{bmatrix} a_1 & a_2 & \cdots & a_n \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix} = \begin{bmatrix} a_1b_1 + a_2b_2 + \cdots + a_nb_n \end{bmatrix}$$

one entry

Example 2.1.5.

$$\begin{bmatrix} -1 & 0 & 3 & 2 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \\ 4 \\ -1 \end{bmatrix} = \begin{bmatrix} (-1)(2) + (0)(3) + (3)(4) + (2)(-1) \end{bmatrix} = \begin{bmatrix} 8 \end{bmatrix}$$

$-2 + 12 \quad -2$

Larger Matrices

If A is an $n \times p$ matrix and B is a $p \times m$ matrix then the **matrix product** of A and B, AB, is an $n \times m$ matrix whose i^{th} row and j^{th} column entry is the real number obtained from multiplying the i^{th} of A by the j^{th} column of B.

THE MUMBER OF COLUMNS OF A MUST BE THE SAME AS THE NUMBER OF ROWS OF B.

Example 2.1.6. $\begin{bmatrix} -1 & 1 \\ 2 & 3 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} -1 & 0 & 3 & -2 \\ 1 & 2 & 2 & 0 \end{bmatrix}$

$$= \begin{bmatrix} \begin{bmatrix} -1 & 1 \end{bmatrix}\begin{bmatrix} -1 \\ 1 \end{bmatrix} & \begin{bmatrix} -1 & 1 \end{bmatrix}\begin{bmatrix} 0 \\ 2 \end{bmatrix} & \begin{bmatrix} -1 & 1 \end{bmatrix}\begin{bmatrix} 3 \\ 2 \end{bmatrix} & \begin{bmatrix} -1 & 1 \end{bmatrix}\begin{bmatrix} -2 \\ 0 \end{bmatrix} \\ \begin{bmatrix} 2 & 3 \end{bmatrix}\begin{bmatrix} -1 \\ 1 \end{bmatrix} & \begin{bmatrix} 2 & 3 \end{bmatrix}\begin{bmatrix} 0 \\ 2 \end{bmatrix} & \begin{bmatrix} 2 & 3 \end{bmatrix}\begin{bmatrix} 3 \\ 2 \end{bmatrix} & \begin{bmatrix} 2 & 3 \end{bmatrix}\begin{bmatrix} -2 \\ 0 \end{bmatrix} \\ \begin{bmatrix} 1 & 0 \end{bmatrix}\begin{bmatrix} -1 \\ 1 \end{bmatrix} & \begin{bmatrix} 1 & 0 \end{bmatrix}\begin{bmatrix} 0 \\ 2 \end{bmatrix} & \begin{bmatrix} 1 & 0 \end{bmatrix}\begin{bmatrix} 3 \\ 2 \end{bmatrix} & \begin{bmatrix} 1 & 0 \end{bmatrix}\begin{bmatrix} -2 \\ 0 \end{bmatrix} \end{bmatrix}$$

$$= \begin{bmatrix} 1+1 & 0+2 & -3+2 & 2+0 \\ -2+3 & 0+6 & 6+6 & -4+0 \\ -1+0 & 0+0 & 3+0 & -2+0 \end{bmatrix} = \begin{bmatrix} 2 & 2 & -1 & 2 \\ 1 & 6 & 12 & -4 \\ -1 & 0 & 3 & -2 \end{bmatrix}$$

Example 2.1.7. $A = \begin{bmatrix} 3 & 2 & 0 \\ -1 & 4 & -6 \end{bmatrix}$ $\quad B = \begin{bmatrix} 5 & -2 \\ 1 & 3 \end{bmatrix}$

Find BA and AB

$$BA = \begin{bmatrix} 5 & -2 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} 3 & 2 & 0 \\ -1 & 4 & -6 \end{bmatrix} = \begin{bmatrix} 15+2 & 10-8 & 0+12 \\ 3-3 & 2+12 & 0-18 \end{bmatrix}$$

$$= \begin{bmatrix} 17 & 2 & 12 \\ 0 & 14 & -18 \end{bmatrix}$$

$$AB = \begin{bmatrix} 3 & 2 & 0 \\ -1 & 4 & -6 \end{bmatrix}\begin{bmatrix} 5 & -2 \\ 1 & 3 \end{bmatrix} \qquad \cancel{3(5)+2(1)+0(?)}$$

NO, can't be done.

4

$\frac{1}{2}(2x) = 7\left(\frac{1}{2}\right)$

$2x\left(\frac{1}{2}\right) = \frac{1}{2}(7)$

$B[Ax] = B[b]$

Multiplication must be on the same side.

Properties of Matrix Multiplication

Theorem 2.1. Let A be an $m \times n$ matrix, and let B and C have sizes for which the indicated sums and products are defined.

a. $A(BC) = (AB)C$ (associative law of multiplication)
b. $A(B+C) = AB + AC$ (left distributive law)
c. $(B+C)A = BA + CA$ (right distributive law)
d. $r(AB) = (rA)B = A(rB)$ for any scalar r
e. $I_m A = A = A I_n$ (Identity for matrix multiplication)

Example 2.1.8. $A = \begin{bmatrix} 2 & -3 \end{bmatrix}$ $\quad B = \begin{bmatrix} 8 & 4 \end{bmatrix}$ $\quad C = \begin{bmatrix} 5 & -2 \end{bmatrix}$ $\quad D = \begin{bmatrix} 15 & 9 \end{bmatrix}$

e. $I_m A = A = A I_n$ (identity for matrix multiplication)

Example 2.1.8. $A = \begin{bmatrix} 2 & -3 \\ -4 & 6 \end{bmatrix}$ $B = \begin{bmatrix} 8 & 4 \\ 5 & 5 \end{bmatrix}$ $C = \begin{bmatrix} 5 & -2 \\ 3 & 1 \end{bmatrix}$ $D = \begin{bmatrix} 15 & 9 \\ 10 & 6 \end{bmatrix}$

Find AC, AB, BA and AD

$$AC = \begin{bmatrix} 2 & -3 \\ -4 & 6 \end{bmatrix}\begin{bmatrix} 5 & -2 \\ 3 & 1 \end{bmatrix} = \begin{bmatrix} 10-9 & -4-3 \\ -20+18 & 8+6 \end{bmatrix} = \begin{bmatrix} 1 & -7 \\ -2 & 14 \end{bmatrix}$$

$$AB = \begin{bmatrix} 2 & -3 \\ -4 & 6 \end{bmatrix}\begin{bmatrix} 8 & 4 \\ 5 & 5 \end{bmatrix} = \begin{bmatrix} 16-15 & 8-15 \\ -32+30 & -16+30 \end{bmatrix} = \begin{bmatrix} 1 & -7 \\ -2 & 14 \end{bmatrix}$$

$$BA = \begin{bmatrix} 8 & 4 \\ 5 & 5 \end{bmatrix}\begin{bmatrix} 2 & -3 \\ -4 & 6 \end{bmatrix} = \begin{bmatrix} 16-16 & -24+24 \\ 10-20 & -15+30 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ -10 & 15 \end{bmatrix}$$

$$AD = \begin{bmatrix} 2 & -3 \\ -4 & 6 \end{bmatrix}\begin{bmatrix} 15 & 9 \\ 10 & 6 \end{bmatrix} = \begin{bmatrix} 30-30 & 18-18 \\ -60+60 & -36+36 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

2.1.4 The Transpose of a Matrix

Definition 2.2. Give an $m \times n$ matrix A, the **transpose** of A is the $n \times m$ matrix, denoted by A^T whose columns are formed by the corresponding rows of A

Properties of Matrix Transpose

Theorem 2.2. Let A and B have sizes for which the indicated sums and products are defined.

a. $(A^T)^T = A$
b. $(A+B)^T = A^T + B^T$
c. For any scalar r, $(rA)^T = rA^T$
d. $(AB)^T = B^T A^T$

Example 2.1.9. $A = \begin{bmatrix} 1 & -3 \\ -2 & 4 \end{bmatrix}$ $\quad x = \begin{bmatrix} 5 \\ 3 \end{bmatrix}$ $\quad B = \begin{bmatrix} 1 & -3 & 7 \\ -2 & 4 & -2 \end{bmatrix}$

Find $(Ax)^T$, $x^T A^T$, xx^T, and $x^T x$.

Can you calculate $A^T x^T$, $(AB)^T$, $A^T B^T$, $B^T A^T$ and, if so, what is the result?

$$(Ax)^T = \left(\begin{bmatrix} 1 & -3 \\ -2 & 4 \end{bmatrix}\begin{bmatrix} 5 \\ 3 \end{bmatrix}\right)^T = \begin{bmatrix} -4 \\ 2 \end{bmatrix}^T = \begin{bmatrix} -4 & 2 \end{bmatrix} \checkmark$$

$$x^T A^T = \begin{bmatrix} 5 & 3 \end{bmatrix}\begin{bmatrix} 1 & -2 \\ -3 & 4 \end{bmatrix} = \begin{bmatrix} -4 & 2 \end{bmatrix} \checkmark$$

$$x x^T = \begin{bmatrix} 5 \\ 3 \end{bmatrix}\begin{bmatrix} 5 & 3 \end{bmatrix} = \begin{bmatrix} 25 & 15 \\ 15 & 9 \end{bmatrix}$$

$$x^T x = \begin{bmatrix} 5 & 3 \end{bmatrix}\begin{bmatrix} 5 \\ 3 \end{bmatrix} = \begin{bmatrix} 34 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & -3 \\ -2 & 4 \end{bmatrix}$$

$$B = \begin{bmatrix} 1 & -3 & 7 \\ -2 & 4 & -2 \end{bmatrix}$$

6

$$A^T x^T = \begin{bmatrix} 1 & -2 \\ -3 & 4 \end{bmatrix}\begin{bmatrix} 5 & 3 \end{bmatrix} = \text{DNE}$$

$$(AB)^T = \left(\begin{bmatrix} 1 & -3 \\ -2 & 4 \end{bmatrix}\begin{bmatrix} 1 & -3 & 7 \\ -2 & 4 & -2 \end{bmatrix}\right)^T = \begin{bmatrix} 7 & -15 & 13 \\ -10 & 22 & -22 \end{bmatrix}^T = \begin{bmatrix} 7 & -10 \\ -15 & 22 \\ 13 & -22 \end{bmatrix}$$

$$A^T B^T = \begin{bmatrix} 1 & -2 \\ -3 & 4 \end{bmatrix}\begin{bmatrix} 1 & -2 \\ -3 & 4 \\ 7 & -2 \end{bmatrix} = \text{DNE}$$

$$B^T A^T = \begin{bmatrix} 1 & -2 \\ -3 & 4 \\ 7 & -2 \end{bmatrix}\begin{bmatrix} 1 & -2 \\ -3 & 4 \end{bmatrix} = (AB)^T$$