Section 1.9

Monday, September 12, 2022 1:42 PM

0 is the additive identity

$0 + 7 = 7$

1 is the multiplicative identity

$1(7) = 7$

Identity function

$f(x) = x$

operation is composition of functions.

$e_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

1.9 The Matrix of a Linear Transformation

Identity Matrix and Basis

Definition 1.14. The **identity matrix** I_n in the $n \times n$ matrix with 1's on the main diagonal and zeros everywhere else.

$$I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 7 \\ 5 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}7 + \begin{bmatrix} 0 \\ 1 \end{bmatrix}5 = \begin{bmatrix} 7 \\ 5 \end{bmatrix}$$

A **basis** is a minimal spanning set. (a more detailed definition shows up later).

The **Standard Basis** for $\mathbb{R}^n$ is the set of vectors $e_i = \begin{bmatrix} 0 & \cdots & 0 & 1 & 0 & \cdots & 0 \end{bmatrix}$ where the 1 is in the i^{th} position.

↑ i^{th} position

$\begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$

If you know what a transformation does to the basis elements you know what it does to all vectors.

$$T(c_1\mathbf{e_1} + c_2\mathbf{e_2} + \cdots + c_n\mathbf{e_n}) = c_1T(\mathbf{e_1}) + c_2T(\mathbf{e_2}) + \cdots + c_nT(\mathbf{e_n})$$

span $\{e_1, e_2, \ldots, e_n\}$

Example 1.9.1. For $\mathbb{R}^2$ the standard basis is $B = \left\{\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}\right\}$. If $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in \mathbb{R}^2$ then $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}x_1 + \begin{pmatrix} 0 \\ 1 \end{pmatrix}x_2$. Use the definition to write $T\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = T\left(\begin{pmatrix} 1 \\ 0 \end{pmatrix}x_1 + \begin{pmatrix} 0 \\ 1 \end{pmatrix}x_2\right)$

$$\boxed{= x_1 T\begin{pmatrix} 1 \\ 0 \end{pmatrix} + x_2 T\begin{pmatrix} 0 \\ 1 \end{pmatrix}}$$

What is the basis for $\mathbb{R}^3$?

standard basis

$$\left\{\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}\right\}$$

$$\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

4 basis elements

Example 1.9.2. Suppose $T : \mathbb{R}^2 \to \mathbb{R}^4$ and we know $T(e_1) = \begin{bmatrix} 3 \\ 4 \\ 5 \\ 1 \end{bmatrix}$ and $T(e_2) = \begin{bmatrix} -3 \\ 2 \\ 5 \\ -7 \end{bmatrix}$. Find a matrix A that has the same mapping.

2 basis elements

$$T = \begin{bmatrix} 3 & -3 \\ 4 & 2 \\ 5 & 5 \\ 1 & -7 \end{bmatrix} \qquad \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{bmatrix} 3 \\ 4 \\ 5 \\ 1 \end{bmatrix}(1) + \begin{bmatrix} -3 \\ 2 \\ 5 \\ -7 \end{bmatrix}0 = \begin{bmatrix} 3 \\ 4 \\ 5 \\ 1 \end{bmatrix}$$

$[T(e_1) \quad T(e_2)]$

Example 1.9.3. Assume that T is a linear transformation. Find the standard matrix of T where T first performs a horizontal shear that transforms e_2 into $e_2 + 12e_1$ (leaving e_1 unchanged) and then reflects points through the line $x_2 = -x_1$.

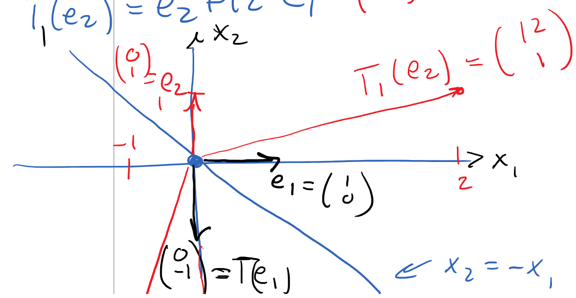
$T_1(e_1) = e_1$

$T_1(e_2) = e_2 + 12e_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} + 12\begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$T(e_1) = \begin{pmatrix} 0 \\ -1 \end{pmatrix}$

$T(e_2) = \begin{pmatrix} -1 \\ -12 \end{pmatrix}$

$$T(x) = \begin{bmatrix} 0 & -1 \\ -1 & -12 \end{bmatrix}$$



$T(e_2) \neq \begin{pmatrix} -1 \\ -12 \end{pmatrix}$

$\begin{pmatrix} 0 \\ -1 \end{pmatrix} = T(e_1)$

$x_2 = -x_1$

-12

$T(e_2) = \begin{pmatrix} -1 \\ -12 \end{pmatrix}$

One-to-one and Onto mappings

Definition 1.15. A mapping $T : \mathbb{R}^n \to \mathbb{R}^m$ is said to be **onto** $\mathbb{R}^m$ if each b in $\mathbb{R}^m$ is the image of at least one x in $\mathbb{R}^n$

A **one-to-one transformation** is a transformation where each x in the domain is mapped to exactly one element in the range. In other words, $x \longmapsto Ax$ is a unique map.

$y = x^2$ $\quad x=2 \quad y=4$ $\quad x=-2 \quad y=4$

Domain $\mathbb{R}$

Range $\quad y \geq 0$

Not onto the real numbers

Example 1.9.4. $A = \begin{bmatrix} 1 & -5 & -7 \\ -3 & 7 & 5 \end{bmatrix}$ is not a one-to-one map because the vector $x = \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix} + \begin{bmatrix} -3 \\ -2 \\ 1 \end{bmatrix} t$ maps to $\begin{bmatrix} -2 \\ -2 \end{bmatrix}$. (see **Example 1.8.4**)

$A\begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -2 \\ -2 \end{bmatrix}$ & $A\begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} -2 \\ -2 \end{bmatrix}$

one-to-one $\quad$ (1-to-1)

y's are unique

one-to-one and onto theorems

Theorem 1.9. Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation, and let A be the standard matrix for T. Then:

No free variables

1. T is one-to-one if and only if the equation $T(x) = 0$ has only the trivial solution.
2. T is one-to-one if and only if the equation $Ax = 0$ has only the trivial solution.
3. T maps $\mathbb{R}^n$ onto $\mathbb{R}^m$ if and only if the columns of A span $\mathbb{R}^m$.
4. T is one-to-one if and only if the columns of A are linearly independent.

Example 1.9.5. $T(x) : \mathbb{R}^3 \to \mathbb{R}^2$ is defined by the matrix $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$. Show that the transformation is **onto** $\mathbb{R}^2$. (Hint: Need to show that you can get to any vector in $\mathbb{R}^2$ (what does that look like?) from some vector in $\mathbb{R}^3$ (what does that look like?)).

Vector in $\mathbb{R}^2$ $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ $\begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}$ vector in $\mathbb{R}^3$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} y_1 + \begin{pmatrix} 0 \\ 1 \end{pmatrix} y_2 + \begin{pmatrix} 0 \\ 0 \end{pmatrix} y_3$$

$$= \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \leftarrow \text{any vector in } \mathbb{R}^2$$

Example 1.9.6. $T(x) : \mathbb{R}^3 \to \mathbb{R}^2$ is defined by the matrix $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$. Show that the transformation is **NOT** one-to-one. (Hint: Need to show that you can get to some vector in $\mathbb{R}^2$ from multiple vectors in $\mathbb{R}^3$ (what does that look like?)).

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \\ 1 \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

two different inputs give same output not 1 to 1

Example 1.9.7. Show that T is a linear transformation by finding a matrix that implements the mappings

1. $T(x_1, x_2, x_3, x_4) = (\underbrace{x_1 + 8x_2}_{1}, \underbrace{0}_{2}, \underbrace{x_1 - 5x_2 + 6x_4}_{3}, \underbrace{x_2 - 3x_3}_{4})$ ← 4 rows because 4 entries in the vector, 4 colums because 4 variables
2. $T(x_1, x_2, x_3) = (\underbrace{x_1 - 5x_2 + 6x_3}_{\text{row 1}}, \underbrace{x_2 - 3x_3}_{\text{row 2}})$ ← 2 rows b/c 2 entries in the vector, 3 colums b/c 3 variables

1. $$\begin{bmatrix} 1 & 8 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & -5 & 0 & 6 \\ 0 & 1 & -3 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} x_1 + \begin{pmatrix} 8 \\ 0 \\ -5 \\ 1 \end{pmatrix} x_2 + \begin{pmatrix} 0 \\ 0 \\ 0 \\ -3 \end{pmatrix} x_3 + \begin{pmatrix} 0 \\ 0 \\ 6 \\ 0 \end{pmatrix} x_4 = \begin{bmatrix} x_1 + 8x_2 \\ 0 \\ x_1 - 5x_2 + 6x_4 \\ x_2 - 3x_3 \end{bmatrix}$$

2. $$\begin{bmatrix} 1 & -5 & 6 \\ 0 & 1 & -3 \end{bmatrix}$$