Section 1.8

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1.8 Introduction to Linear Transformations

$Ax = b$ is a matrix equation. We can also think of the matrix A as doing something to the vector x. We say that A "acts" on x by multiplication. This produces a new vector Ax.

Definition 1.12. A **Transformation** (or **Function** or **Mapping**) T from $\mathbb{R}^n$ to $\mathbb{R}^m$ is a rule that assigns to each vector x in $\mathbb{R}^n$ a vector $T(x)$ in $\mathbb{R}^m$. The set $\mathbb{R}^n$ is called the **Domain** of T, and $\mathbb{R}^m$ is called the **Codomain** of T. The notation

$$T : \mathbb{R}^n \to \mathbb{R}^m$$

T is represented by a matrix A

indicates that the domain of T is $\mathbb{R}^n$ and the codomain is $\mathbb{R}^m$.

For x in $\mathbb{R}^n$, the vector $T(x)$ in $\mathbb{R}^m$ is called the **image** of x. The set of all images is the **Range** of T.
The transformation $T(x) = Ax$ is sometimes written as $x \longmapsto Ax$ and

start the domain is $\mathbb{R}^n$

$\longmapsto$ "maps to"

end the codomain is $\mathbb{R}^m$

the range is $\{T(x) : x \in \mathbb{R}^n\}$

$y = x^2$

Domain $\mathbb{R}$

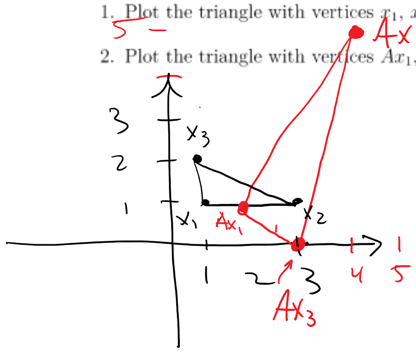
Codomain $\mathbb{R}$

Range $y \geq 0$

$\lim_{x \to 0}$

Example 1.8.1. Given $A = \begin{bmatrix} 1 & 1 \\ 2 & -1 \end{bmatrix}$, $x_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $x_2 = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$, $x_3 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$.

1. Plot the triangle with vertices x_1, x_2, x_3.
2. Plot the triangle with vertices Ax_1, Ax_2, Ax_3.



$$Ax_1 = \begin{bmatrix} 1 & 1 \\ 2 & -1 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} 1 + \begin{bmatrix} 1 \\ -1 \end{bmatrix} 1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

$$Ax_2 = \begin{bmatrix} 1 & 1 \\ 2 & -1 \end{bmatrix}\begin{bmatrix} 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} 3 + \begin{bmatrix} 1 \\ -1 \end{bmatrix} 1 = \begin{bmatrix} 4 \\ 5 \end{bmatrix}$$

$$Ax_3 = \begin{bmatrix} 1 & 1 \\ 2 & -1 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} 1 + \begin{bmatrix} 1 \\ -1 \end{bmatrix} 2 = \begin{bmatrix} 3 \\ 0 \end{bmatrix}$$

Matrix multiplication. $A = [a_1 \ a_2 \ \cdots \ a_n]$ columns of A are a_i

Vector $b = \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix}$

$$Ab = a_1 b_1 + a_2 b_2 + \cdots + a_n b_n \leftarrow$$

$$Ab = a_1 b_1 + a_2 b_2 + \dots + a_n b_n \leftarrow$$

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \end{bmatrix} 1 + \begin{bmatrix} 2 \\ 5 \end{bmatrix} 1 + \begin{bmatrix} 3 \\ 6 \end{bmatrix} (1) = \begin{pmatrix} 6 \\ 15 \end{pmatrix}$$

$\mathbb{R}^3$ vector

$\mathbb{R}^2$

Definition 1.13. A transformation T is **Linear** if

1. $T(\mathbf{u} + \mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})$ for all $\mathbf{u}$ and $\mathbf{v}$ in the domain of T:
2. $T(c\mathbf{u}) = cT(\mathbf{u})$ for all scalars c and all $\mathbf{u}$ in the domain of T.

$T(0) = 0$

$T(c\mathbf{u} + d\mathbf{v}) = T(cu) + T(dv) = cT(u) + dT(v)$

Example 1.8.2. Define $T(x) = Ax$ where $A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & -1 & 2 & 3 \end{bmatrix}$. Find the transformation $x \longmapsto Ax$ of $x = \begin{bmatrix} 0 \\ 1 \\ 2 \\ 3 \end{bmatrix}$. What are the domain and codomain of A.

$$T\left(\begin{pmatrix} 0 \\ 1 \\ 2 \\ 3 \end{pmatrix}\right) = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & -1 & 2 & 3 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 \\ 5 \end{bmatrix} 0 + \begin{pmatrix} 2 \\ -1 \end{pmatrix} 1 + \begin{pmatrix} 3 \\ 2 \end{pmatrix} 2 + \begin{pmatrix} 4 \\ 3 \end{pmatrix} 3 = \begin{pmatrix} 20 \\ 12 \end{pmatrix}$$

Domain $\mathbb{R}^4$

Co domain $\mathbb{R}^2$

Example 1.8.3. Find all the vectors that map onto $\begin{bmatrix} -2 \\ -2 \end{bmatrix}$ given the matrix transformation defined by $A = \begin{bmatrix} 1 & -5 & -7 \\ -3 & 7 & 5 \end{bmatrix}$. Need to solve $Ax = \begin{bmatrix} -2 \\ -2 \end{bmatrix}$

$$\left[\begin{array}{ccc|c} 1 & -5 & -7 & -2 \\ -3 & 7 & 5 & -2 \end{array}\right] \xrightarrow{3R_1+R_2 \to R_2} \left[\begin{array}{ccc|c} 1 & -5 & -7 & -2 \\ 0 & -8 & -16 & -8 \end{array}\right]$$

$$-\frac{1}{8}R_2 \left[\begin{array}{ccc|c} 1 & -5 & -7 & -2 \\ 0 & 1 & 2 & 1 \end{array}\right] \xrightarrow{R_1+5R_2 \to R_1} \left[\begin{array}{cccc} 1 & 0 & 3 & 3 \\ 0 & 1 & 2 & 1 \end{array}\right]$$

zero

$$x_1 + 3x_3 = 3$$
$$x_2 + 2x_3 = 1$$

$$x_1 = 3 - 3x_3$$
$$x_2 = 1 - 2x_3$$
$$x_3 = x_3$$

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix} + \begin{bmatrix} -3 \\ -2 \\ 1 \end{bmatrix} t$$