Section 1.7

Friday, September 2, 2022 1:45 PM

1.7 Linear Independence

Definition 1.11. A set of vectors $\{v_1, v_2, \ldots, v_n\}$ in $\mathbb{R}^n$ is said to be **Linearly Dependent** if there exists a set of constants $c_1, c_2, \ldots, c_n$, not all zero, such that

$$c_1v_1 + c_2v_2 + \cdots + c_nv_n = 0.$$

Ex: $\begin{bmatrix}1\\0\end{bmatrix}$ $\begin{bmatrix}2\\0\end{bmatrix}$

$2\begin{bmatrix}1\\0\end{bmatrix} - \begin{bmatrix}2\\0\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix}$

The set is **Linearly Independent** otherwise.

Example 1.7.1. Are the vectors $v_1 = \begin{bmatrix}1\\-3\\0\end{bmatrix}$, $v_2 = \begin{bmatrix}0\\1\\2\end{bmatrix}$, $v_3 = \begin{bmatrix}4\\-6\\12\end{bmatrix}$ linearly independent? If possible find a dependence relation among them.

Dependent

nonzero constants that add up to 0

$c_1v_1 + c_2v_2 + c_3v_3 = 0$?

$$\left[\begin{array}{ccc|c} 1 & 0 & 4 & 0 \\ -3 & 1 & -6 & 0 \\ 0 & 2 & 12 & 0 \end{array}\right] \xrightarrow{RR} \begin{bmatrix} 1 & 0 & 4 & 0 \\ 0 & 1 & 6 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$c_1 \quad c_2 \quad c_3 = 0$

$c_1 + 4c_3 = 0$

$c_2 + 6c_3 = 0$

$c_3 = c_3$

$$\begin{bmatrix}c_1\\c_2\\c_3\end{bmatrix} = \begin{bmatrix}-4\\-6\\1\end{bmatrix} c_3$$

$\boxed{-4v_1 - 6v_2 + v_3 = 0}$

Sometimes we talk about the linear independence of the matrix columns. ← same question as ex 1.7.1

Theorem 1.5. The columns of matrix $A = [a_1 \ a_2 \ \cdots \ a_n]$ are linearly dependent if $Ax = 0$ has a nontrivial solution.

Example 1.7.2. Determine if the columns of matrix $A = \begin{bmatrix} 0 & 0 & -3 \\ 0 & 5 & 4 \\ 2 & -8 & 1 \end{bmatrix}$ are linearly dependent.

$$\begin{bmatrix} 0 & 0 & -3 & 0 \\ 0 & 5 & 4 & 0 \\ 2 & -8 & 1 & 0 \end{bmatrix} \xrightarrow{R_1 \leftrightarrow R_3} \begin{bmatrix} 2 & -8 & 1 & 0 \\ 0 & 5 & 4 & 0 \\ 0 & 0 & -3 & 0 \end{bmatrix}$$

Consistent & has only the zero solution

$$\xrightarrow{RREF} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$c_1 = 0$

$c_2 = 0$

$c_3 = 0$

Linearly Independent

Theorem 1.6. Some characterizations of linearly dependent sets:

1. One vector is always independent
2. Two vectors are dependent if one is a multiple of the other.
3. A set of vectors $\{v_1, v_2, \ldots, v_n\}$ is linearly dependent if at least one can be written as a multiple of the others.

$$v_i = c_1 v_1 + \cdots + c_{i-1} v_{i-1} + c_{i+1} v_{i+1} + \cdots + c_n v_n.$$

Example 1.7.3. Let $u = \begin{bmatrix} 3 \\ 2 \\ -4 \end{bmatrix}$, $v = \begin{bmatrix} -6 \\ 1 \\ 7 \end{bmatrix}$, $w = \begin{bmatrix} 0 \\ -5 \\ 2 \end{bmatrix}$, $z = \begin{bmatrix} 3 \\ 7 \\ -5 \end{bmatrix}$.

one vector written as a multiple of the others.

(a) Is w a linear combination of u, v, and z? → solve $x_1 u + x_2 v + x_3 z \overset{?}{=} w$

Is the set $\{u, v, w, z\}$ linearly independent?

$$\left[\begin{array}{ccc|c} 3 & -6 & 3 & 0 \\ 2 & 1 & 7 & -5 \\ -4 & 7 & -5 & 2 \end{array}\right] \xrightarrow{\frac{1}{3}R_1} \begin{bmatrix} 1 & -2 & 1 & 0 \\ 2 & 1 & 7 & -5 \\ -4 & 7 & -5 & 2 \end{bmatrix} \xrightarrow[\cancel{R_3+4R_1}]{R_2-2R_1} \begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 5 & 5 & -5 \\ 0 & -1 & -1 & 2 \end{bmatrix}$$

$$\xrightarrow[-R_3]{R_2+5R_3} \begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 0 & 0 & 15 \\ 0 & 1 & 1 & -2 \end{bmatrix}$$

No answer so w is NOT a combination of u, v & z.

$$x_1 u + x_2 v + x_3 z + x_4 w = 0 \quad \left[\begin{array}{cccc|c} 1 & -2 & 1 & 0 & 0 \\ 0 & 1 & 1 & -2 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right]$$

x_3 is a free variable so ∞ solutions so Dependent

Theorem 1.7. If a set contains more vectors than there are entries in each vector then the set is linearly dependent.

Theorem 1.8. If a set $S = \{v_1, v_2, \ldots, v_n\}$ contains the zero vector it is linearly dependent.

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \end{bmatrix} \qquad 0\begin{bmatrix} 1 \\ 0 \end{bmatrix} + 7\begin{bmatrix} 0 \\ 0 \end{bmatrix} = 0$$

Example 1.7.4. Each statement is either true (in all cases) or false (for at least one example). If false, construct a specific example, called a counterexample, to show that the statement is not always true. If true, give a justification, not just a specific example

1. The columns of a matrix A are linearly independent if the equation $Ax = 0$ has the trivial solution.
   False $Ax=0$ always has $x=0$ as solution.

2. If S is a linearly dependent set, then each vector is a linear combination of the other vectors in S.
   False by example 1.7.3

3. The columns of a 4 x 5 matrix are linearly dependent.
   True $\begin{bmatrix} x & x & x & x & x \\ x & x & x & x & x \\ x & x & x & x & x \\ x & x & x & x & x \end{bmatrix}$ more vectors (5) than entries in the vector (4) so Dependent.

4. If x and y are linearly independent and if $\{x, y, z\}$ is linearly dependent, then z is in Span$\{x, y\}$.
   Span(x,y) = $c_1 x + c_2 y = 0$ when c_1 & c_2 are zero So $c_1 x + c_2 y + c_3 z = 0$ c_1, c_2 & c_3 are not all zero. & z is a combination of x & y

5. If v_1 and v_2 are in $\mathbb{R}^4$ and v_2 is not a scalar multiple of v_1, then $\{v_1, v_2\}$ is linearly independent.
   True If two vectors are dependent then $v_2 = c v_1$

6. If $v_1, \ldots, v_4$ are in $\mathbb{R}^4$ and v_3 is **not** a linear combination of v_1, v_2, v_4, then $\{v_1, v_2, v_3, v_4\}$ is linearly independent.
   False only one has to be, not all of them.

7. If $v_1, \ldots, v_4$ are linearly independent vectors in $\mathbb{R}^4$, then $\{v_1, v_2, v_3\}$ is also linearly independent.
   True $c_1 v_1 + c_2 v_2 + c_3 v_3 + c_4 v_4 = 0$ only when $c_1 = c_2 = c_3 = c_4 = 0$ also true for any subset of $\{v_1, v_2, v_3, v_4\}$

Example 1.7.5. Determine by inspection if the given set is linearly independent.

1. $\left\{ \begin{bmatrix} 5 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 8 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \end{bmatrix} \right\}$ Dependent, more vectors then entries

2. $\left\{ \underset{v_1}{\begin{bmatrix} 4 \\ -2 \\ 6 \end{bmatrix}}, \underset{v_2}{\begin{bmatrix} 6 \\ -3 \\ 9 \end{bmatrix}} \right\}$ $\frac{3}{2} v_1 = v_2$

3. $\left\{ \begin{bmatrix} 3 \\ 5 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 6 \\ 5 \\ 4 \end{bmatrix} \right\}$ Dependent. b/c 0 vector.

Example 1.7.6. Justify your answers.

1. For what values of h is v_3 in Span$\{v_1, v_2\}$? $c_1 v_1 + c_2 v_2 = v_3$ (solve that)
2. For what values of h is $\{v_1, v_2, v_3\}$ linearly dependent?

$$v_1 = \begin{bmatrix} 1 \\ -5 \\ -3 \end{bmatrix}, \qquad v_2 = \begin{bmatrix} -2 \\ 10 \\ 6 \end{bmatrix}, \qquad v_3 = \begin{bmatrix} 2 \\ -9 \\ h \end{bmatrix}$$

same question