Section 1.5

Friday, September 2, 2022 1:45 PM

1.5 Solution Sets of Linear Systems

Definition 1.10. A system of linear equations is said to be **homogeneous** if it can be written in the form $Ax = 0$. The **trivial solution** is the solution $x = 0$.

Theorem 1.3. The trivial solution is ALWAYS as solution to the homogeneous equation $Ax = 0$. The homogeneous equation $Ax = 0$ has a nontrivial solution if and only if the equation has at least one free variable.

Example 1.5.1. $\begin{bmatrix} 1 & 0 & 4 \\ -3 & 1 & -6 \\ 0 & 2 & 12 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$$\left[\begin{array}{ccc|c} 1 & 0 & 4 & 0 \\ -3 & 1 & -6 & 0 \\ 0 & 2 & 12 & 0 \end{array}\right] \xrightarrow[R_3 - 2R_2 \to R_3]{R_2 + 3R_1 \to R_2} \left[\begin{array}{cccc} 1 & 0 & 4 & 0 \\ 0 & 1 & 6 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \begin{array}{l} \leftarrow x_1 + 4x_3 = 0 \\ \leftarrow x_2 + 6x_3 = 0 \\ \leftarrow 0 = 0 \end{array}$$

$$x_1 = -4x_3 = -4t$$
$$x_2 = -6x_3 = -6t$$
$$x_3 = x_3 = t$$

Free variable

$t = 1$ $(-4, -6, 1)$ is a solution

Example 1.5.2. Write the general solution of $x_1 + 9x_2 - 4x_3 = 7$

$$x_1 = 7 - 9x_2 + 4x_3 = 7 - 9t + 4r$$

2 Free variables $\begin{cases} x_2 = x_2 = t \\ x_3 = x_3 = r \end{cases}$

$$\boxed{x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 7 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} -9 \\ 1 \\ 0 \end{bmatrix} t + \begin{bmatrix} 4 \\ 0 \\ 1 \end{bmatrix} r}$$

Constants

Vector form of the solution.

1.5.1 Nonhomogeneous solutions

Example 1.5.3. Describe all solutions of $Ax = b$ where

$$A = \begin{bmatrix} 1 & 0 & 4 \\ -3 & 1 & -6 \\ 0 & 2 & 12 \end{bmatrix} \quad \text{and} \quad b = \begin{bmatrix} 4 \\ -3 \\ 18 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 4 & 4 \\ -3 & 1 & -6 & -3 \\ 0 & 2 & 12 & 18 \end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix} 1 & 0 & 4 & 4 \\ 0 & 1 & 6 & 9 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$x_1 + 4x_3 = 4 \qquad x_1 = 4 - 4t$$
$$x_2 + 6x_3 = 9 \qquad x_2 = 9 - 6t$$
$$x_3 = x_3 \qquad x_3 = 0 + t$$

homogeneous solution

Vectors

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 4 \\ 9 \\ 0 \end{bmatrix} + \boxed{\begin{bmatrix} -4 \\ -6 \\ 1 \end{bmatrix} t}$$

slope

particular solution

$$Ax = \underbrace{A\begin{bmatrix} 4 \\ 9 \\ 0 \end{bmatrix}}_{b} + \underbrace{A\begin{bmatrix} -4 \\ -6 \\ 1 \end{bmatrix}}_{0} t$$

Theorem 1.4. Suppose the equation $Ax = b$ is consistent for some given b, and let p be a solution. Then the solution set of $Ax = b$ is the set of all vectors of the from $w = p + v_h$ where v_h is any solution of the homogeneous equation $Ax = 0$.

homogeneous.

any solution $\left\{ \begin{array}{l} \text{particular} \\ \text{solution} \end{array} \right.$

Nonhomogeneous

$y = 3x + 5$

$y - 3x = 5$

Homogeneous

$y = 3x$

$y - 3x = 0$

Same slope

$$\text{Vector } \overrightarrow{pq} = q - p = \begin{bmatrix} 0 \\ -2 \end{bmatrix} - \begin{bmatrix} -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ -4 \end{bmatrix}$$

Find a parametric equation of the line M through **p** and **q** for the given values of **p** and **q**. [Hint: M is parallel to the vector **q** − **p** shown in the figure.]

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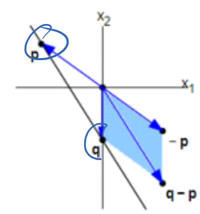

$$\mathbf{p} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}, \mathbf{q} = \begin{bmatrix} 0 \\ -2 \end{bmatrix}$$

Need $\overset{p}{\text{point}}$ & $\overset{v}{\overbrace{\text{vector in the}}}$ direction of the line

line = p + v · t

↑ parameter

$\mathbf{x} = \square + t\square$ (t in $\mathbb{R}$)

$$\begin{bmatrix} 0 \\ -2 \end{bmatrix} + t \begin{bmatrix} 1 \\ -4 \end{bmatrix}$$

2 x 5 matrix

$$\left[\begin{array}{ccccc|c} 1 & 0 & \square & \square & \square & b_1 \\ 0 & 1 & \square & \square & \square & b_2 \end{array}\right]$$

$\underbrace{\square \ \square \ \square}_{\text{3 free variables}}$