Section 1.4

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1.4 The Matrix Equation $Ax = b$

1.4.1 Matrix multiplication

Definition 1.9. If A is the matrix with columns $a_1, a_2, a_3, \dots, a_n$ and x is in $\mathbb{R}^n$ the the **product** Ax is defined as

$$Ax = \begin{bmatrix} a_1 & a_2 & \cdots & a_n \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = a_1x_1 + a_2x_2 + \cdots + a_nx_n$$

Example 1.4.1. Multiply $\begin{array}{cc} a_1 & a_2 \end{array}$ $\begin{bmatrix} -4 & 2 \\ 1 & 6 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 6 \end{bmatrix} \begin{array}{l} \leftarrow x_1 \\ \leftarrow x_2 \end{array} = 2\begin{bmatrix} -4 \\ 1 \\ 0 \end{bmatrix} + 6\begin{bmatrix} 2 \\ 6 \\ 1 \end{bmatrix}$

$$= \begin{bmatrix} -8 \\ 2 \\ 0 \end{bmatrix} + \begin{bmatrix} 12 \\ 36 \\ 6 \end{bmatrix} = \begin{bmatrix} 4 \\ 38 \\ 6 \end{bmatrix}$$

Example 1.4.2. Find Ax given that $A = \begin{bmatrix} 1 & -3 & 5 \\ 0 & 2 & -3 \end{bmatrix}$ and $x = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$

$$Ax = \begin{bmatrix} 1 \\ 0 \end{bmatrix}(1) + 2\begin{bmatrix} -3 \\ 2 \end{bmatrix} + 3\begin{bmatrix} 5 \\ -3 \end{bmatrix} = \begin{bmatrix} 10 \\ -5 \end{bmatrix}$$

1.4.2 Three ways to write the system of equations $Ax = b$

1. Write $Ax = b$ explicitly in the the form

$$\begin{bmatrix} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \cdots + a_{1n}x_n = & b_1 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \cdots + a_{2n}x_n = & b_2 \\ \vdots & \vdots \\ a_{m1}x_1 + a_{m2}x_2 + a_{m3}x_3 + \cdots + a_{mn}x_n = & b_m \end{bmatrix}$$

Method 2

2. Write as a vector equation (Linear combination of column vectors)

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n = b \leftarrow$$

b is a linear combination of the columns of A.

where $a_i = \begin{bmatrix} a_{1i} \\ a_{2i} \\ \vdots \\ a_{mi} \end{bmatrix}$

3. Write as an augmented matrix

$$\left[\overbrace{a_1 \quad a_2 \quad \cdots \quad a_n}^{\text{coefficients matrix}} \,\middle|\, b \right]$$

$\nwarrow$ constants.

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3. Write as an augmented matrix

$$\left[\begin{array}{cccc|c} \multicolumn{4}{c|}{\overbrace{a_1 \quad a_2 \quad \cdots \quad a_n}^{\text{coefficients matrix}}} & b \end{array}\right]$$

constants.

Theorem 1.1. Let A be an $m \times n$ matrix and b be a column vector in $\mathbb{R}^m$. Then the following statements are logically equivalent. That is, for a particular A, either they are all true statements or all false.

1. The equation $Ax = b$ has a solution
2. b is a linear combination of the columns of A
3. b is in span$\{a_1, a_2, \ldots, a_n\}$
4. $Ax = b$ is consistent.

$$\begin{array}{c} \\ \\ \end{array}\left[\begin{array}{ccc|c} 1 & 0 & 2 & 1 \\ 0 & 1 & 3 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right] \quad b$$

For different b could get

$$\left[\begin{array}{ccc|c} 1 & 0 & 2 & a \\ 0 & 1 & 3 & b \\ 0 & 0 & 0 & \text{Not zero} \end{array}\right]$$

Theorem 1.2. Let A be an $m \times n$ matrix. Then the following statements are logically equivalent. That is, for a particular A, either they are all true statements or all false.

1. For each b in $\mathbb{R}^m$, The equation $Ax = b$ has a solution
2. Each b in $\mathbb{R}^m$ is a linear combination of the columns of A
3. The columns of A span $\mathbb{R}^m$.
4. A has a pivot position in every row.

$$\left[\begin{array}{c|c} A & b \end{array}\right]$$

can't have a row of zeros otherwise there are elements in $\mathbb{R}^m$ that have no solution for $Ax = b$

Example 1.4.3. Write as a vector equation and as matrix equation

$$\begin{aligned} x_1 - x_3 &= 5 \\ -2x_1 + x_2 + 2x_3 &= -6 \\ 2x_2 + 2x_3 &= -4 \end{aligned}$$

Matrix

$$\left[\begin{array}{ccc|c} 1 & 0 & -1 & 5 \\ -2 & 1 & 2 & -6 \\ 0 & 2 & 2 & -4 \end{array}\right]$$

vector

$$\begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix} x_1 + \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} x_2 + \begin{bmatrix} -1 \\ 2 \\ 2 \end{bmatrix} x_3 = \begin{bmatrix} 5 \\ -6 \\ -4 \end{bmatrix}$$

$X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

Example 1.4.4. Let $\vec{u} = \begin{bmatrix} 0 \\ 4 \\ 4 \end{bmatrix}$ and $A = \begin{bmatrix} 3 & -5 \\ -2 & 6 \\ 1 & 1 \end{bmatrix}$. Is $\vec{u}$ spanned by the columns of A? solve $Ax = u$

make zero

$$\begin{array}{cc} x_1 & x_2 \end{array}$$

$$\left[\begin{array}{cc|c} 3 & -5 & 0 \\ -2 & 6 & 4 \\ 1 & 1 & 4 \end{array}\right] \xrightarrow{R_1 \text{ to top}} \left[\begin{array}{cc|c} 1 & 1 & 4 \\ 3 & -5 & 0 \\ -2 & 6 & 4 \end{array}\right] \xrightarrow[R_3+2R_1 \to R_3]{R_2-3R_1 \to R_2} \begin{bmatrix} 1 & 1 & 4 \\ 0 & -8 & -12 \\ 0 & 8 & 12 \end{bmatrix} \xrightarrow{R_3+R_2} \left[\begin{array}{cc|c} 1 & 1 & 4 \\ 0 & -8 & -12 \\ 0 & 0 & 0 \end{array}\right]$$

$$\begin{array}{cccl} 3 & -5 & 0 & R_2 \\ -3 & -3 & -12 & -3R_1 \\ \hline 0 & -8 & -12 & R_2 \end{array}$$

Consistent so has solution

Example 1.4.5. Given $A = \begin{bmatrix} 1 & -3 & -4 \\ -3 & 2 & 6 \\ 5 & -1 & -8 \end{bmatrix}$ answer the following.

1. How many rows of A contain a pivot position? 2
2. Does the equation $Ax = b$ have a solution for each $b \in \mathbb{R}^3$? NO
3. Can each vector in $\mathbb{R}^3$ be written as a linear combination of the columns of matrix A? NO
4. Do the columns of A span $\mathbb{R}^3$? NO

$$\begin{bmatrix} 1 & -3 & -4 \\ -3 & 2 & 6 \\ 5 & -1 & -8 \end{bmatrix} \xrightarrow[R_3-5R_1 \to R_3]{3R_1+R_2 \to R_2} \begin{bmatrix} 1 & -3 & -4 \\ 0 & -7 & -6 \\ 0 & 14 & 12 \end{bmatrix} \xrightarrow{R_3+2R_2} \begin{bmatrix} 1 & -3 & -4 \\ 0 & -7 & -6 \\ 0 & 0 & 0 \end{bmatrix}$$

Example 1.4.6. Construct a 3×3 matrix, not in echelon form, whose columns do NOT span $\mathbb{R}^3$. Show that your matrix has the desired property.

$$\begin{bmatrix} 1 & 0 & 0 \\ 2 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

Example 1.4.7. Find a column of the matrix $A = \begin{bmatrix} 12 & -7 & 11 & -9 & 5 \\ -9 & 4 & -8 & 7 & -3 \\ -6 & 11 & -7 & 3 & -9 \\ 4 & -6 & 10 & -5 & 12 \end{bmatrix}$ that can be deleted and yet have the remaining matrix columns span $\mathbb{R}^4$.

Input interpretation

$$\begin{pmatrix} 12 & -7 & 11 & -9 & 5 \\ -9 & 4 & -8 & 7 & -3 \end{pmatrix}$$

Example 1.4.7. Find a column of the matrix $A = \begin{bmatrix} -6 & 11 & -7 & 3 & -9 \\ 4 & -6 & 10 & -5 & 12 \end{bmatrix}$ that can be deleted and yet have the remaining matrix columns span $\mathbb{R}^4$.

don't need column 4

No Pivot

Input interpretation

row reduce	$\begin{pmatrix} 12 & -7 & 11 & -9 & 5 \\ -9 & 4 & -8 & 7 & -3 \\ -6 & 11 & -7 & 3 & -9 \\ 4 & -6 & 10 & -5 & 12 \end{pmatrix}$

Result

$$\begin{pmatrix} 1 & 0 & 0 & -\frac{10}{21} & 0 \\ 0 & 1 & 0 & -\frac{25}{84} & 0 \\ 0 & 0 & 1 & -\frac{41}{84} & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

Dimensions