Section 1.3

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1.3 Vector Equations

Notation:

$\mathbb{R}$ is the real numbers

$\mathbb{R}^2$ is $\mathbb{R} \times \mathbb{R}$ the xy-plane

$\mathbb{R}^3$ is 3D space.

Definition 1.6. A **vector** is an ordered list of numbers.... for now.

$$\text{Column Vector: } v = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ \vdots \\ v_n \end{bmatrix} \qquad \text{Row Vector: } v = \begin{bmatrix} v_1 & v_2 & v_3 & \cdots & v_n \end{bmatrix}$$

Example 1.3.1. $\vec{w}_1 = \begin{bmatrix} 1 \\ -5 \end{bmatrix}$ is a vector in $\mathbb{R}^2$. ← 2 x 1

v $\vec{v}$

$\vec{w}_2 = \underline{(1, -5)} = \underline{< 1, -5 >}$ is a vector in $\mathbb{R}^2$

$\vec{w}_3 = \begin{bmatrix} 1 & -5 \end{bmatrix}$ is a vector in $\mathbb{R}^2$ ← 1 x 2

What is the difference between $\vec{w}_1$, $\vec{w}_2$, and $\vec{w}_3$?

Mostly notation but

W_1 & W_3 are not the same because 2x1 vs. 1x2

A **scalar multiple** of vector $\vec{v}$ is the vector $c\vec{v}$ obtained by multiplying every element in vector $\vec{v}$ by scalar c. For example

Scalar is a single number

$$2 * \vec{w}_1 = 2 * \begin{bmatrix} 1 \\ -5 \end{bmatrix} = \begin{bmatrix} 2 \\ -10 \end{bmatrix}$$

20 is a scalar

[20] is a vector

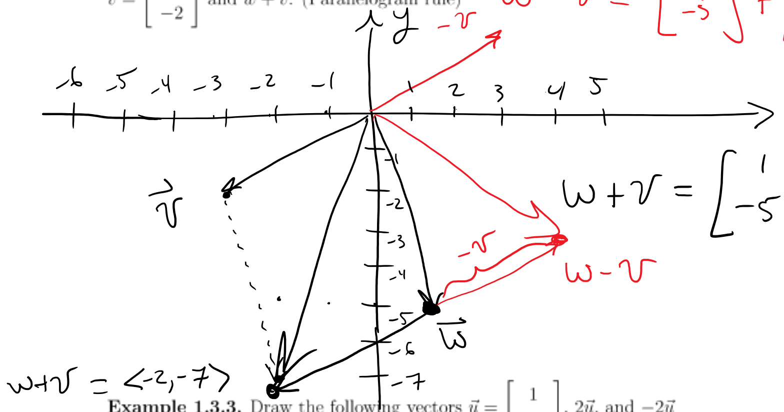
Adding vectors is done by adding the corresponding coordinates. For example

$$\begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ \vdots \\ v_n \end{bmatrix} + \begin{bmatrix} w_1 \\ w_2 \\ w_3 \\ \vdots \\ w_n \end{bmatrix} = \begin{bmatrix} v_1 + w_1 \\ v_2 + w_2 \\ v_3 + w_3 \\ \vdots \\ v_n + w_n \end{bmatrix}$$

$\mathbb{R}^2 \rightarrow \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ $\qquad$ $\begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} \leftarrow \mathbb{R}^3$

Example 1.3.2. Vectors can be represented as arrows in the plane. Graph $\vec{w} = \begin{bmatrix} 1 \\ -5 \end{bmatrix}$ and $\vec{v} = \begin{bmatrix} -3 \\ -2 \end{bmatrix}$ and $\vec{w} + \vec{v}$. (Parallelogram rule)

$$w - v = \begin{bmatrix} 1 \\ -5 \end{bmatrix} + \begin{bmatrix} 3 \\ 2 \end{bmatrix}$$

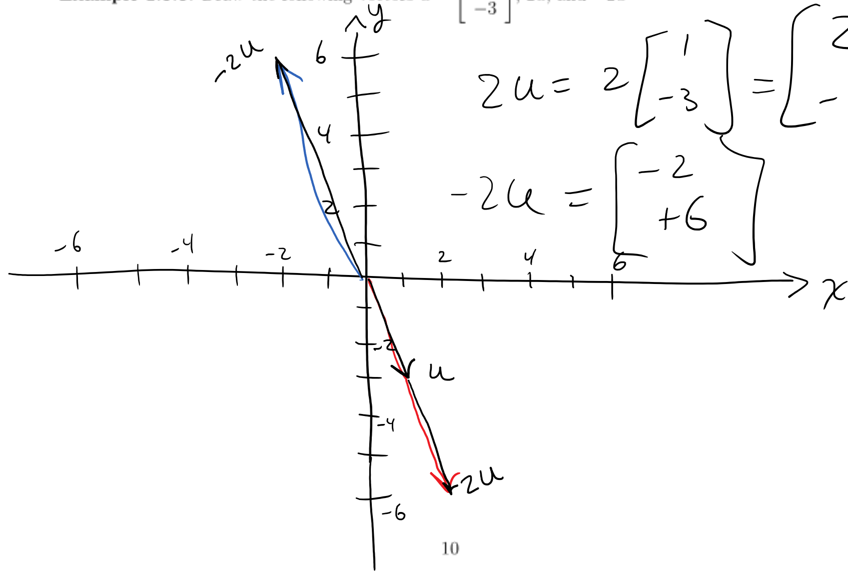


$$w + v = \begin{bmatrix} 1 \\ -5 \end{bmatrix} + \begin{bmatrix} -3 \\ -2 \end{bmatrix} = \begin{bmatrix} -2 \\ -7 \end{bmatrix}$$

Example 1.3.3. Draw the following vectors $\vec{u} = \begin{bmatrix} 1 \\ -3 \end{bmatrix}$, $2\vec{u}$, and $-2\vec{u}$

$$2u = 2\begin{bmatrix} 1 \\ -3 \end{bmatrix} = \begin{bmatrix} 2 \\ -6 \end{bmatrix}$$

$$-2u = \begin{bmatrix} -2 \\ +6 \end{bmatrix}$$



1.3.1 Algebraic Properties of Vectors in $\mathbb{R}^n$.

For all u, v, and w in $\mathbb{R}^n$ and all scalars a and b:

$$0 = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}$$

1. $u + v = v + u$
2. $(u + v) + w = u + (v + w)$
3. $u + 0 = 0 + u = u$
4. $u + (-u) = -u + u = u - u = 0$
5. $a(u + v) = au + av$
6. $(a + b)u = au + bu$
7. $(ab)u = a(bu) = b(au)$
8. $1u = u$

1.3.2 Linear Combinations of Vectors

Definition 1.7. Given vectors $v_1, v_2, v_3, \ldots, v_n$ and constants $c_1, c_2, c_3, \ldots, c_n$ the vector

$$y = c_1v_1 + c_2v_2 + c_3v_3 + \cdots + c_nv_n$$

is called a **Linear Combination** of the vectors v_i with weights c_i.

Example 1.3.4. Given $\vec{v}_1 = \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix} -3 \\ 1 \\ 8 \end{bmatrix}$, $y_1 = \begin{bmatrix} 1 \\ -5 \\ -3 \end{bmatrix}$, and $y_2 = \begin{bmatrix} -\frac{7}{2} \\ -5 \\ -3 \end{bmatrix}$, are y_1 or y_2 a linear combination of $\vec{v}_1$ and $\vec{v}_2$.

Solution: If y_1 is a linear combination of $\vec{v}_1$ and $\vec{v}_2$ then y_1 must solve the following equation for some values of x_1 and x_2:

$$y_1 = x_1\vec{v}_1 + x_2\vec{v}_2$$

y_1

$$\begin{bmatrix} 1 \\ -5 \\ -3 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix} + x_2 \begin{bmatrix} -3 \\ 1 \\ 8 \end{bmatrix}$$

x_1 & x_2 are constants.

That is a system of equations and we can solve it using an augmented matrix and row reduction:

Make zero Make zero

$$\begin{array}{c} \left[\begin{array}{cc|c} 1 & -3 & 1 \\ 0 & 1 & -5 \\ -2 & 8 & -3 \end{array}\right] \\ x_1 \quad x_2 \end{array} \xrightarrow{R_3 + 2R_1 \to R_3} \begin{bmatrix} 1 & -3 & 1 \\ 0 & 1 & -5 \\ 0 & 2 & -1 \end{bmatrix}$$

$$\xrightarrow{R_3 - 2R_2 \to R_3} \left[\begin{array}{cc|c} 1 & -3 & 1 \\ 0 & 1 & -5 \\ 0 & 0 & 9 \end{array}\right] \leftarrow 0x_1 + 0x_2 = 9$$

$0 = 9$ No

y_1 is Not a linear combination

$0 = 4$ No

y_1 is Not a linear combination of v_1 & v_2

$y_2 = \begin{bmatrix} -7/2 \\ -5 \\ -3 \end{bmatrix}$ so solve $\begin{bmatrix} 1 & -3 & -7/2 \\ 0 & 1 & -5 \\ -2 & 8 & -3 \end{bmatrix} \xrightarrow{RR} \begin{array}{c} \begin{array}{ccc} x_2 & x_2 & \end{array} \\ \begin{bmatrix} 1 & 0 & -\frac{37}{2} \\ 0 & 1 & -5 \\ 0 & 0 & 0 \end{bmatrix} \end{array}$

$$\boxed{x_1 = -\frac{37}{2} \quad x_2 = -5}$$

$$\boxed{y_2 = -\frac{37}{2}v_1 - 5v_2}$$

yes, Linear combination.

1.3.3 Spanning sets

$\hat{i} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ $\hat{j} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ $\hat{k} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

Definition 1.8. If $v_1, v_2, v_3, \ldots, v_p$ are vectors in $\mathbb{R}^n$ then the set of all linear combinations of $v_1, v_2, v_3, \ldots, v_p$ is called the **Span of** $v_1, v_2, v_3, \ldots, v_p$ represented $\text{span}\{v_1, v_2, v_3, \ldots, v_p\}$ and is the subset of $\mathbb{R}^n$ spanned by $v_1, v_2, v_3, \ldots, v_p$.

$$\text{span}\{v_1, v_2, v_3, \ldots, v_p\} = c_1v_1 + c_2v_2 + c_3v_3 + \cdots + c_pv_p$$

$\hat{i}, \hat{j}, \hat{k}$ span $\mathbb{R}^3$

The previous example could have been stated as: Is y_1 or y_2 in $\text{span}\{v_1, v_2\}$

$a\hat{i} + b\hat{j} + c\hat{k}$

Example 1.3.5. Determine if $b = \begin{bmatrix} 11 \\ -5 \\ 9 \end{bmatrix}$ is in the span of the column vectors that form

$$A = \begin{bmatrix} 1 & -2 & -6 \\ 0 & 3 & 7 \\ 1 & -2 & 5 \end{bmatrix} = [v_1 \;\; v_2 \;\; v_3]$$

$$\begin{array}{ccc} \uparrow & \uparrow & \uparrow \\ v_1 & v_2 & v_3 \end{array}$$

$$\left\{ v_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix},\; v_2 = \begin{bmatrix} -2 \\ 3 \\ -2 \end{bmatrix},\; v_3 = \begin{bmatrix} -6 \\ 7 \\ 5 \end{bmatrix} \right\}$$

$$c_1 v_1 + c_2 v_2 + c_3 v_3 = b$$

$$\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} c_1 + \begin{bmatrix} -2 \\ 3 \\ -2 \end{bmatrix} c_2 + \begin{bmatrix} -6 \\ 7 \\ 5 \end{bmatrix} c_3 = \begin{bmatrix} 11 \\ -5 \\ 9 \end{bmatrix}$$

$$\begin{array}{c} \left[\begin{array}{ccc|c} 1 & -2 & -6 & 11 \\ 0 & 3 & 7 & -5 \\ 1 & -2 & 5 & 9 \end{array}\right] \\ \begin{array}{ccc} c_1 & c_2 & c_3 \end{array} \end{array} \xrightarrow{R_3 - R_1 \to R_3} \begin{array}{c} \begin{bmatrix} 1 & -2 & -6 & 11 \\ 0 & 3 & 7 & -5 \\ 0 & 0 & 11 & -2 \end{bmatrix} \\ \begin{array}{ccc} \uparrow & \uparrow & \uparrow \\ c_1 & c_2 & c_3 \end{array} \end{array}$$

$$\cancel{x_1}\, c_1 - 2\cancel{x_2}\, c_2 - 6\cancel{x_3}\, c_3 = 11$$

$$\cancel{x_2}\, c_2 = \text{some \#}$$

$$3\cancel{x_2}\, c_2 + 7\cancel{x_3}\, c_3 = -15$$

$$\cancel{x_3}\, c_3 = -2/11$$

yes, there is a solution so

b is in the span of $\{v_1, v_2, v_3\}$

span of columns of A