Understand these concepts from Chapter 1:

1. Elementary row operations. (How many are there? What are they?)
2. How to use elementary row operations to reduce a matrix.
3. The **forward phase** and **backward phase** of row reduction.
4. The **coefficient matrix** and the **augmented matrix** of a system of equations.
5. The difference between **echelon form** and **reduced echelon form** (and what these forms are useful for).
6. What are **leading entries** and **pivot columns**?
7. How to tell if a system is consistent.
8. How to tell if a system has a unique solution.
9. What are **basic variables** and **free variables**?
10. How the number of free variables affects the "shape" of a solution set.
11. What is a **linear combination** of vectors?
12. How to write a system of equations as a **vector equation** and as a **matrix equation**.
13. How to write the solution to a system in **parametric vector form**.
14. What is a **homogeneous system** and what special features does it have?
15. Changing coefficients in a system affects the span of a solution, and changing constants in a system affects the basepoint.
16. What relation exists between the solution set to a homogeneous system and to a nonhomogeneous system if they have the same coefficient matrix?
17. How to determine if a set of vectors is **linearly independent**.
18. How to determine if a single vector b is in the **span** of a set of vectors.
19. How to determine if a set of vectors in $\mathbb{R}^n$ span all of $\mathbb{R}^n$.
20. What operations does a **linear transformation** respect?
21. How to construct the **standard matrix** of a linear transformation.
22. How to sketch the results of a linear transformation and determine whether it is a shear, a reflection, a rotation, a dilation, etc. (I won't test you on the specific terminology, but you should be able to describe what happens.)
23. How to tell whether a linear transformation is **one-to-one** or **onto**.

Understand these concepts from Chapter 2:

1. Addition and Subtraction with Matrices
2. Multiplication of a matrix by a scalar
3. Multiplication of two matrices
4. Calculate the Transpose of a Matrix
5. Calculate the Inverse of a Matrix
6. Characterizations of Invertible Matrices (the list is long)

   Let A be a square $n \times n$ matrix. Then the following statements are equivalent.

   a. A is an invertible matrix.
   b. A is row equivalent to I_n
   c. A has n pivot positions.
   d. The equation $Ax = 0$ has only the trivial solution.
   e. The columns of A form a linearly independent set.
   f. The linear transfromation $x \mapsto Ax$ is one-to-one.
   g. The equation $Ax = b$ has at least one solution for each b in $\mathbb{R}^n$.
   h. The columns of A span $\mathbb{R}^n$.
   i. The linear transformation $x \mapsto Ax$ maps $\mathbb{R}^n$ onto $\mathbb{R}^n$.
   j. There is an $n \times n$ matrix C such that $CA = I$.
   k. There is an $n \times n$ matrix D such that $AD = I$.
   l. A^T is an invertible matrix.

1. Use elementary operations to convert matrix A to reduced echelon form. Indicate in your work at which step you have completed the forward phase of the reduction. Write the matrix you have obtained in the space provided.

$$A = \begin{bmatrix} 1 & 4 & 1 & -2 \\ 2 & 8 & -1 & -1 \\ 1 & 3 & 1 & 0 \\ 0 & -1 & 0 & 2 \end{bmatrix} \qquad \text{reduced echelon form} \begin{bmatrix} \_\_\_ & \_\_\_ & \_\_\_ & \_\_\_ \\ \_\_\_ & \_\_\_ & \_\_\_ & \_\_\_ \\ \_\_\_ & \_\_\_ & \_\_\_ & \_\_\_ \\ \_\_\_ & \_\_\_ & \_\_\_ & \_\_\_ \end{bmatrix}$$

Answer the following questions about the column vectors of A by circling the correct response:

The column vectors of A are linearly (a) dependent (b) independent.

The column vectors of A (a) span $\mathbb{R}^4$ (b) do not span $\mathbb{R}^4$.

Suppose A was the augmented matrix of a system of equations. Answer the following questions by circling the correct response:

The system is (a) consistent (b) inconsistent (c) Can't be determined.

The system has (a) zero (b) one (c) two free variables.

The column vectors of the coefficient matrix (a) span $\mathbb{R}^3$ (b) do not span $\mathbb{R}^3$.

Now suppose A is the matrix of a linear transformation $T(x) = Ax$. Answer the following questions by circling the correct response:

$T(x)$ maps (a) $\mathbb{R}^3 \mapsto \mathbb{R}^4$ (b) $\mathbb{R}^3 \mapsto \mathbb{R}^3$ (c) $\mathbb{R}^4 \mapsto \mathbb{R}^3$ (d) $\mathbb{R}^4 \mapsto \mathbb{R}^4$

The transformation $T(x)$ (a) is onto the codomain (b) is not onto the codomain.

Now suppose A is the coefficent matrix of a homogeneous system of equations. Answer the following questions by circling the correct response:

The system is (a) consistent (b) inconsistent (c) can't be determined

The system has (a) zero (b) one (c) two free variables.

If the system is consistent the solution set forms (a) a single point (b) a line (c) a plane.

2. Three of the following statements are equivalent (whenever one of them is true, the others are also true), but one is not. Circle the statement which is not equivalent to the others.

   (a) The column vectors of A are a linearly dependent set.
   (b) A is invertible.
   (c) The equation Ax = b has a free variable.
   (d) The equation Ax = 0 has non-trivial solutions.

3. Classify the following statements as true or false. If a statement is false, give a counterexample or a brief explanation why.
   (a) If a system of linear equations has no free variables, then it must have a unique solution.
   (b) If a system $Ax = b$ has more than one solution, then so does the system $Ax = 0$.
   (c) If w is a linear combination of u and v, then u is a linear combination of v and w.
   (d) If the columns of A span $\mathbb{R}^n$, then the system $Ax = b$ is consistent for all b in $\mathbb{R}^n$.
   (e) If $\{v_1, v_2, v_3\}$ is a linearly independent set of vectors, then so is $\{v_1, v_2\}$.
   (f) If the homogeneous equation Ax = 0 has a non-trivial solution, then the linear transformation T(x) = Ax is onto.
   (g) If A and B are $m \times n$ matrices. then both $A^T B$ and AB^T are defined.
   (h) If $AB = C$ and C has 2 columns, then A has 2 columns.
   (i) If $BC = BD$, then $C = D$.
   (j) If A and B are square matrices, then $(A + B)(AB) = A^2B^2$.
   (k) If $AB = I$, then A is invertible.
   (l) The set of pivot columns of a matrix is linearly independent.

4. Give an example of each of the following, or state that no such example exists.
   (a) A set of three linearly independent vectors in $\mathbb{R}^4$.
   (b) A vector v such that $\text{Span}\left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix}, v \right\} = \mathbb{R}^2$.
   (c) A linear combination of $v = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$ and $w = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$.

5. The matrix $\begin{bmatrix} 1 & -4 & 0 & 0 & 7 & 2 \\ 0 & 0 & 1 & 0 & 1 & 5 \\ 0 & 0 & 0 & 1 & -2 & 3 \end{bmatrix}$ represents the augmented matrix of a system of linear equations. Express the solution set of this system in vector form. Use t, s, and/or r as needed to represent free variables in the system.

6. Determine the value(s) of h such that the matrix $\begin{bmatrix} 1 & 4 & -2 \\ 3 & h & -8 \end{bmatrix}$ is the augmented matrix of a consistent linear system.

7. Rewrite the matrix equation $\begin{bmatrix} 2 & 1 & 3 \\ 1 & 1 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 5 \\ 12 \end{bmatrix}$ as an equation involving a linear combination of vectors in $\mathbb{R}^2$. (You do not have to solve this equation.)

8. Compute the inverse of $\begin{bmatrix} 7 & 5 \\ 3 & 2 \end{bmatrix}$. (Note: This is a 2×2 matrix so there are two ways you can do this!)

9. Given that $\begin{bmatrix} 2 & 0 & b \\ 1 & 4 & -2 \end{bmatrix} \begin{bmatrix} 5 & 1 \\ 3 & 3 \\ -1 & 4 \end{bmatrix} = \begin{bmatrix} 9 & 6 \\ 19 & a \end{bmatrix}$ what can you say about the values of a and b?

10. Any problems from the homework.

Answers

1.

$$\text{Row Reduced } A = \begin{bmatrix} 1 & 0 & 0 & 7 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Answer the following questions about the column vectors of A by circling the correct response:

The column vectors of A are linearly (a) dependent

The column vectors of A (b) do not span $\mathbb{R}^4$.

Suppose A was the augmented matrix of a system of equations. Answer the following questions by circling the correct response:

The system is (a) consistent

The system has (a) zero

The column vectors of the coefficient matrix (b) do not span $\mathbb{R}^3$.

Now suppose A is the matrix of a linear transformation $T(x) = Ax$. Answer the following questions by circling the correct response:

$T(x)$ maps (d) $\mathbb{R}^4 \mapsto \mathbb{R}^4$

The transformation $T(x)$ (b) is not onto the codomain.

Now suppose A is the coefficent matrix of a homogeneous system of equations. Answer the following questions by circling the correct response:

The system is (a) consistent

The system has (b) one.

The solution set forms (b) a line.

2. (b) A is invertible.

3. Classify the following statements as true or false. If a statement is false, give a counterexample or a brief explanation why.

   (a) F: $\begin{bmatrix} A & \vert b \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$

   (b) T

   (c) T

   (d) T

   (e) T

   (f) F: $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} x \neq \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

   (g) T

   (h) F: B has 2 columns.

   (i) F: $B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, $C = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $D = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

   (j) F: $AB \neq BA$.

   (k) T

   (l) T

4. Give an example of each of the following, or state that no such example exists.

   (a) $\{(1,0,0,0),(0,1,0,0),(0,0,1,0)\}$

   (b) A vector $v = (0,1)$.

   (c) $\begin{bmatrix} 1 \\ -2 \end{bmatrix} + \begin{bmatrix} 3 \\ 2 \end{bmatrix}$.

5. $$\begin{aligned} x_1 &= 2 + 4t - 7s \\ x_2 &= t \\ x_3 &= 5 - r \\ x_4 &= 3 + 2r \\ x_5 &= r \end{aligned}$$

6. $h \neq 12$

7. $\begin{bmatrix} 2 \\ 1 \end{bmatrix} x_1 + \begin{bmatrix} 1 \\ 1 \end{bmatrix} x_2 + \begin{bmatrix} 3 \\ 4 \end{bmatrix} x_3 = \begin{bmatrix} 5 \\ 12 \end{bmatrix}$

8. $A^{-1} = \begin{bmatrix} -2 & 5 \\ 3 & -7 \end{bmatrix}$.

9. $a = 5$, $b = 1$