Contents

6.1 Inner Product, Length, and Orthogonality

6.1.1 Multiplying Vectors

There are two ways to multiply vectors u and v:

1. the **cross product** $u \times v$. We will not be discussing cross products in this class.
2. the **dot product** $u \cdot v$. The dot product is also called the **inner product**.

Inner Product

Definition 6.1. The **inner product** (or **dot product**) of two $x \times 1$ vectors u and v

$$u = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix} \qquad v = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$$

is the product $u \cdot v = u^{\mathrm{T}} v$.

$$u \cdot v = \begin{bmatrix} u_1 & u_2 & \cdots & u_n \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} = u_1 v_1 + u_2 v_2 + \cdots + u_n v_n$$

IMPORTANT: Notice that when you take the dot product of two vectors you have a scalar answer. The dot product is a single number.

Example 6.1.1. Find $u \cdot v$ for $u = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ and $v = \begin{bmatrix} -1 \\ 2 \\ -7 \end{bmatrix}$

Properties of the Inner Product

Properties of the Inner Product: Let u, v, and w be vectors in $\mathbb{R}^n$, and let c be a scalar. Then

1. $u \cdot v = v \cdot u$
2. $(u + v) \cdot w = u \cdot w + v \cdot w$
3. $(cu) \cdot v = c(u \cdot v) = u \cdot (cv)$
4. $u \cdot u \geq 0$, and $u \cdot u = 0 \Leftrightarrow u = 0$

Properties 2 and 3 can be combined into

$$(c_1 u_1 + \cdots + c_p u_p) \cdot w = c_1(u_1 \cdot w) + \cdots + c_p(u_p \cdot w)$$

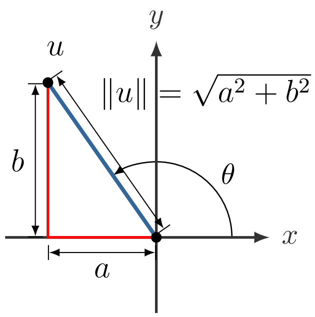
6.1.2 Length of a Vector

Length of a Vector

Definition 6.2. The **length** (or **norm**) of a vector v is the nonnegative scalar $\|v\|$ defined by

$$\|v\| = \sqrt{v \cdot v} = \sqrt{v_1^2 + v_2^2 + \cdots + v_n^2}, \qquad \text{and } \|v\|^2 = v \cdot v$$

A **unit vector** is a vector of length 1.
If v is a vector the unit vector in the direction of v is $\dfrac{v}{\|v\|}$.
The process of changing a vector v into a unit vector is called **normalizing** v.



Example 6.1.2. For $v =< -1, 2, -7 >$

1. Find the length of v.
2. Find a unit vector in the direction of v.
3. Write v as (magnitude) $\cdot$ (direction) where the direction is a unit vector.

Example 6.1.3. Find the dot product between $u =< 12, 3, -5 >$ and $v =< 2, -3, 3 >$

Orthogonal Vectors

Definition 6.3. Two vectors u and v are **orthogonal (perpendicular)** to each other if $u \cdot v = 0$.

Theorem 6.1. Two vectors u and v are orthogonal if and only if $\|u+v\|^2 = \|u\|^2 + \|v\|^2$. (ie. the Pythagorean theorem is true)

6.2 Orthogonal Sets

Orthogonal Sets

Definition 6.4. A set of vectors $\{v_1, v_2, \ldots, v_n\}$ is an **orthogonal set** if each pair of distinct vectors is orthogonal. ie:

$$v_i \cdot v_j = 0 \qquad \text{for each } i \neq j$$

Definition 6.5. An **orthogonal basis** is a basis that is also an orthogonal set.

Definition 6.6. An **orthonormal basis (set)** is an orthogonal basis (set) of unit vectors (length 1).

Theorem 6.2. Orthogonal sets are linearly independent.

Example 6.2.1. Show that the following vectors form an orthogonal set.

$$u_1 =< 1, -2, 1 >, \quad u_2 =< 0, 1, 2 >, \quad u_3 =< -5, -2, 1 >$$

Example 6.2.2. Construct an orthonormal basis from

$$u_1 =< 1, -2, 1 >, \quad u_2 =< 0, 1, 2 >, \quad u_3 =< -5, -2, 1 >$$

6.2.1 Orthogonal Projection of One Vector onto Another

Projection of One Vector onto Another.

Theorem 6.3. The angle θ between two vectors, u and v, can be calculated using the dot product:

$$u \cdot v = \|u\| \|v\| \cos\theta$$

$$\cos\theta = \frac{u \cdot v}{\|u\| \|v\|}$$

Definition 6.7. We can use the angle and the dot product to find the **projection of v onto u.**

$$\hat{v} = \text{proj}_u v = \left(\frac{u \cdot v}{\|u\|^2} \right) u = \left(\frac{u \cdot v}{u \cdot u} \right) u$$

Example 6.2.3. Find the projection of $v = \begin{bmatrix} 5 \\ 6 \end{bmatrix}$ onto

1. $u_1 = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$ and
2. $u_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$
3. What is the sum of the two projections?

Orthogonal Basis

Theorem 6.4. Let $\{u_1, u_2, \dots, u_p\}$ be an orthogonal basis for a subspace W of $\mathbb{R}^n$. For each $y \in W$, the weights in

$$y = c_1 u_1 + c_2 u_2 + \cdots + c_p u_p$$

are given by

$$c_i = \frac{u_i \cdot y}{u_i \cdot u_i}$$

Note: The weights c_i are the projections onto each vector in the orthogonal basis.

Example 6.2.4. Write $y = \begin{bmatrix} 6 \\ 1 \\ -8 \end{bmatrix}$ as a linear combination of

$$u_1 = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}, \qquad u_2 = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}, \qquad u_3 = \begin{bmatrix} -5 \\ -2 \\ 1 \end{bmatrix}$$

We showed in example 6.2.1 that this is an orthogonal basis.

6.3 Orthogonal Projections

Recall: From section 6.2 we saw that we could project one vector $\vec{v}$ onto another vector $\vec{u}$: $\text{proj}_{\vec{u}}\vec{v}$.

Now we would like to project vector $\vec{v}$ onto a subspace W of $\mathbb{R}^n$: $\hat{v} = \text{proj}_W\vec{v}$. $\vec{v}$ can be written as a component in W ($\hat{v}$) and a component perpendicular to W (z).

Definition 6.8. The space perpendicular to $W = \text{span}\{u_1, u_2, \ldots, u_p\}$ is called the **orthogonal complement of** W and is written $W^\perp$ (read "W perpendicular" or simply "W perp"). Note: $\mathbb{R}^n = W + W^\perp$

Example 6.3.1. Show that $W^\perp$ is a subspace of $\mathbb{R}^n$.

Start with $u \in W$ and show

1. zero vector (is $0 \in W^\perp$?)
2. closed under addition. (for $v \in W^\perp$ and $x \in W^\perp$ is $v + x \in W^\perp$?)
3. all other properties inherited from $\mathbb{R}^n$

Orthogonal Basis

Theorem 6.5. Suppose $\{u_1, u_2, \ldots, u_n\}$ is an orthogonal basis for $\mathbb{R}^n$ and $W = \{u_1, u_2, \ldots, u_p\}$ then any vector $y \in \mathbb{R}^n$

$$y = c_1 u_1 + c_2 u_2 + \cdots + c_p u_p + c_{p+1} u_{p+1} + \cdots + c_n u_n$$

are given by

$$c_i = \frac{u_i \cdot y}{u_i \cdot u_i}$$

Note: The weights c_i are the projections onto each vector in the orthogonal basis.

Example 6.3.2. Find the orthogonal projection of y onto the subspace spanned by u_1 and u_2 where

$$y = \begin{bmatrix} -1 \\ 4 \\ 3 \end{bmatrix}, \qquad u_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \qquad u_2 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$$

Find $\hat{y} = \text{proj}_W y = c_1 u_1 + c_2 u_2$

Example 6.3.3. W is the subspace spanned by

$$u_1 = \begin{bmatrix} 1 \\ 3 \\ -2 \end{bmatrix}, \qquad u_2 = \begin{bmatrix} 5 \\ 1 \\ 4 \end{bmatrix}.$$

Write $y = \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix}$ as a sum of a vector in W and a vector in $W^{\perp}$ (ie. $y = \hat{y} + z$)

Example 6.3.4. Find the closest point ($\hat{y}$) and the shortest distance ($||z||$) to $\vec{y} = \begin{bmatrix} 3 \\ 1 \\ 5 \\ 1 \end{bmatrix}$ in the subspace

$$W = \{u_1, u_2\} = \left\{ \begin{bmatrix} 3 \\ 1 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix} \right\}.$$

6.4 The Gram-Schmidt Process

The Gram-Schmidt Process is a technique by which, if you are given any basis for a subspace V, you can calculate an orthogonal basis for that subspace. The key step in the Gram-Schmidt Process is the calculation of the orthogonal projection of a vector $\mathbf{v}$ onto a subspace W, sometimes written as $\hat{\mathbf{v}} = \text{proj}_W \mathbf{v}$:

Orthogonal Projection

Let $\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_p\}$ be an orthogonal set of vectors in $\mathbb{R}^n$ and W be the subspace spanned by these vectors. Let $\mathbf{v}$ be any vector in $\mathbb{R}^n$.
The **orthogonal projection** of $\mathbf{v}$ onto W is given by

$$\widehat{\mathbf{v}} = \frac{\mathbf{v} \cdot \mathbf{u}_1}{\mathbf{u}_1 \cdot \mathbf{u}_1}\mathbf{u}_1 + \frac{\mathbf{v} \cdot \mathbf{u}_2}{\mathbf{u}_2 \cdot \mathbf{u}_2}\mathbf{u}_2 + \cdots + \frac{\mathbf{v} \cdot \mathbf{u}_p}{\mathbf{u}_p \cdot \mathbf{u}_p}\mathbf{u}_p$$

Example 6.4.1. Let $\mathbf{u}_1 = \begin{bmatrix} 2 \\ 5 \\ -1 \end{bmatrix}$, $\mathbf{u}_2 = \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}$, and $\mathbf{v} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ and $W = \text{Span}\{\mathbf{u}_1, \mathbf{u}_2\}$.

Find $\hat{\mathbf{v}} = \text{proj}_W \mathbf{v}$:

The Gram-Schmidt Process

Let $\mathfrak{B} = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p\}$ be any linearly independent set of vectors and let V be the subspace spanned by $\mathfrak{B}$. We'll apply the Gram-Schmidt Process to find an orthogonal (or orthonormal) set of vectors which spans V.

1. We leave the first vector completely unchanged for now. That is, $\mathbf{w}_1 = \mathbf{v}_1$.
2. To find the other vectors, we calculate the projection of $\mathbf{v}_j$ onto the subspace spanned by $\{\mathbf{w}_1, \ldots, \mathbf{w}_{j-1}\}$,
$$\hat{\mathbf{v}}_j = \frac{\mathbf{v}_j \cdot \mathbf{w}_1}{\mathbf{w}_1 \cdot \mathbf{w}_1}\mathbf{w}_1 + \frac{\mathbf{v}_j \cdot \mathbf{w}_2}{\mathbf{w}_2 \cdot \mathbf{w}_2}\mathbf{w}_2 + \cdots + \frac{\mathbf{v}_j \cdot \mathbf{w}_{j-1}}{\mathbf{w}_{j-1} \cdot \mathbf{w}_{j-1}}\mathbf{w}_{j-1}$$
then set $\mathbf{w}_j = \mathbf{v}_j - \hat{\mathbf{v}}_j$. (Optional: You may multiply $\mathbf{w}_j$ by the lowest common denominator of its components if that helps.)
3. The set $\{\mathbf{w}_1, \ldots, \mathbf{w}_p\}$ is an orthogonal basis for W.

If you want an orthonormal basis for W then continue as follows:

4. Once the vectors $\{\mathbf{w}_1, \ldots, \mathbf{w}_p\}$ have been computed, scale them to a length of 1: $\mathbf{u}_j = \dfrac{\mathbf{w}_j}{||\mathbf{w}_j||}$
5. The set $\{\mathbf{u}_1, \ldots, \mathbf{u}_p\}$ is an orthonormal basis for V

Example 6.4.2. Find an orthonormal basis for for $W = \text{Span}\left\{\begin{bmatrix}0\\4\\2\end{bmatrix}, \begin{bmatrix}5\\6\\-7\end{bmatrix}\right\}$

Example 6.4.3. Find an orthogonal basis for Col A where

$$A = \begin{bmatrix} 3 & -5 & 1 \\ 1 & 1 & 1 \\ -1 & 5 & -2 \\ 3 & -7 & 8 \end{bmatrix} \qquad ans = \begin{bmatrix} 3 \\ 1 \\ -1 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \\ 3 \\ -1 \end{bmatrix}, \begin{bmatrix} -3 \\ 1 \\ 1 \\ 3 \end{bmatrix}$$

Example 6.4.4. Find an orthogonal basis for

$$\mathfrak{B} = \text{Span}\left\{ \begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 4 \\ -4 \\ 2 \end{bmatrix}, \begin{bmatrix} 5 \\ -4 \\ -3 \\ 7 \\ 1 \end{bmatrix} \right\} \qquad ans = \begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \\ 3 \\ -3 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 2 \\ 2 \\ -2 \end{bmatrix}$$