Contents

5.1 Eigenvectors and Eigenvalues

For this section we will assume that A is an $n \times n$ matrix. So any transformation $T(x) = Ax$ sends vectors from $\mathbb{R}^n$ to $\mathbb{R}^n$.

Example 5.1.1. Consider the linear transformation $T(x) = Ax$ defined by $A = \begin{bmatrix} 4 & 1 \\ -2 & 1 \end{bmatrix}$.

1. Describe what this transformaton does to the standard basis vectors in $\mathbb{R}^2$

2. Let $\mathfrak{b}_1 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$. Calculate $T(\mathfrak{b}_1)$ and describe what happens.

3. Let $\mathfrak{b}_2 = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$. Calculate $T(\mathfrak{b}_2)$ and describe what happens.

Eigenvectors and Eigenvalues

Definition 5.1. An **eigenvector** of an $n \times n$ matrix A is a nonzero vector x such that $Ax = \lambda x$ for some scalar λ. The scalar λ is called an **eigenvalue** of A if there is a nontrivial solution to the equation $Ax = \lambda x$. (Note that an eigenvector must be nonzero but eignevalues can be zero.)

In this section we will be given either an eigenvalue or an eigenvector for each problem.

Example 5.1.2. Let $A = \begin{bmatrix} 3 & 2 \\ 3 & 8 \end{bmatrix}$. Show that $\begin{bmatrix} 1 \\ 3 \end{bmatrix}$ is an eigenvector of A. Determine the corresponding eigenvalue.

Example 5.1.3. Let $A = \begin{bmatrix} 3 & 2 \\ 3 & 8 \end{bmatrix}$. Show that $\lambda = 2$ is an eigenvalue of A. Determine eigenvector whose eigenvalue is 2.

Eigenspace

Definition 5.2. The set of all solutions to $(A - \lambda I)x = 0$ is a subspace of $\mathbb{R}^n$. It is called the **eigenspace** of A corresponding to the eigenvalue λ.

Example 5.1.4. Let $A = \begin{bmatrix} 6 & 3 & -4 \\ 2 & 7 & -4 \\ 2 & 3 & 0 \end{bmatrix}$. if $\lambda = 4$ is an eigenvalue of A find a basis for the eigenspace of A

Eigenvector Linear Independence Theorem

Theorem 5.1. If $v_1, \ldots, v_r$ are eigenvectors that correspond to distinct eigenvalues $\lambda_1, \ldots, \lambda_r$ of an $n \times n$ matrix A, then the set $\{v_1, \ldots, v_r\}$ is linearly independent.

Example 5.1.5. A is an $m \times n$ matrix. Mark each statement TRUE or FALSE.

1. If $Ax = \lambda x$ for some vector x, then λ is an eigenvalue of A.

2. A matrix A is not invertible if and only if 0 is an eigenvalue of A ($Ax = 0x$).

3. A number c is an eigenvalue of A if and only if the equation $(A - cI)x = 0$ has a nontrivial solution.

4. To find the eigenvalues of A, reduce A to echelon form.

5. If $Ax = \lambda x$ for some scalar λ, then x is an eigenvector of A.

6. If v_1 and v_2 are linearly independent eigenvectors, then they correspond to distinct eigenvalues.

7. An eigenspace of A is a null space of a certain matrix.

8. An $n \times n$ matrix can have at most n eigenvalues.

5.2 The Characteristic Equation

5.2.1 The Eigenvalue Problem

We would like to find solutions to

$$A\vec{x} = \lambda\vec{x}$$

where λ is a constant. In these cases multiplication by the matrix is the same as multiplication by a constant. These constants (λ) are called **eigenvalues** and their associated vectors ($\vec{x}$) are called **eigenvectors**.
So how do we find eigenvalues and eigenvectors?
We want $A\vec{x} = \lambda\vec{x}$ so

$$\begin{aligned} A\vec{x} - \lambda\vec{x} &= 0 \\ (A - \lambda I)\vec{x} &= 0 \end{aligned} \tag{1}$$

We know this has nonzero solutions if and only if

$$\det(A - \lambda I) = 0 \tag{2}$$

Step 1: Solve equation (2) for the eigenvalues λ.
Step 2: The eigenvectors are the solutions to equation (1) for a particular value of λ.

Example 5.2.1. Find all eigenvalues and eigenvectors for $A = \begin{bmatrix} 1 & 3 \\ 4 & 2 \end{bmatrix}$ and find a basis for the eigenspace for the solution.

Step 1: $(A - \lambda I) = \begin{bmatrix} 1 & 3 \\ 4 & 2 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1-\lambda & 3 \\ 4 & 2-\lambda \end{bmatrix}$

$\det(A - \lambda I) = (1-\lambda)(2-\lambda) - 12 = 0$

Step 2: Solve $(A - \lambda I)\vec{x} = 0$ for $\vec{x}$

Characteristic Equation

Definition 5.3. The equation

$$\det(A - \lambda I) = 0$$

is called the **characteristic equation** of the matrix A. A scalar λ is an eigenvalue of an $n \times n$ matrix A if and only if λ is a solution to the characteristic equation of A

Example 5.2.2. Find the characteristic equation of

$$A = \begin{bmatrix} 7 & -2 & -4 & -1 \\ 0 & 1 & 7 & -8 \\ 0 & 0 & 5 & -2 \\ 0 & 0 & 0 & 7 \end{bmatrix}$$

Note that the factor $(7 - \lambda)$ and the eigenvalue $\lambda = 7$ appear twice. The eigenvalue 7 is said to have multiplicity 2.

Example 5.2.3. Find the characteristic equation, the eigenvalues ($\lambda = 3, 3$) and the eigenvectors of

$$A = \begin{bmatrix} 2 & 1 \\ -1 & 4 \end{bmatrix}$$

Example 5.2.4. Find the characteristic equation, the eigenvalues and the eigenvectors of

$$A = \begin{bmatrix} 4 & 0 & 0 \\ 5 & 3 & 2 \\ -2 & 0 & 2 \end{bmatrix}$$

5.2.2 Eigenvalue Properties

Eigenvalue Properties

Some properties of eigenvalues and eigenvectors (Eigenpairs)

1. Eigenvectors are not unique.
2. A matrix can have a zero eigenvalue
3. A real matrix may have one or more complex eigenvalues and eigenvectors.
4. Eigenvectors corresponding to distinct eigenvalues are linearly independent.

The Invertible Matrix Theorem (continued)

The Invertible Matrix Theorem (continued)
Let A be an $n \times n$ matrix. Then the following statements are equivalent to the statements found in the Invertible Matrix Theorem given in Chapters 2 and 4 (including the statement that A is invertible):

s. The number 0 is *not* an eigenvalue of A.

t. The determinant of A is *not* zero.

Example 5.2.5. Find the eigenvalues, eigenvectors and a basis for the eigenspace for

$$A = \begin{bmatrix} 3 & 2 & 4 \\ 2 & 0 & 2 \\ 4 & 2 & 3 \end{bmatrix} \text{ (Partial solution: Basis for } E_{-1} = \{(1, -4, 1), (1, 0, -1)\}\text{)}$$

5.3 Diagonalization

5.3.1 Diagonal matrices

Recall:

The **characteristic equation** of a square matrix A, formally notated $\det(A - \lambda I) = 0$, is the equation obtained by subtracting the variable λ from the entries along the main diagonal of A, then taking the determinant and setting it equal to zero.

The roots of this polynomial equation are the **eigenvalues** of A. If a root is repeated k times, we say that eigenvalue has **algebraic multiplicity** k.

Example 5.3.1. Find the eigenvalues and eigenvectors of $A = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix}$

Let D be a diagonal matrix (a square matrix in which all of the entries are zero, except possibly those on the main diagonal). Then computing powers of D are simple, as this example illustrates:

Example 5.3.2. If $D = \begin{bmatrix} 5 & 0 \\ 0 & 3 \end{bmatrix}$, then $D^2 = \begin{bmatrix} 5 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 5 & 0 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} 25 & 0 \\ 0 & 9 \end{bmatrix}$

In general, $D^k = \begin{bmatrix} 5^k & 0 \\ 0 & 3^k \end{bmatrix}$.

Similar Matrices

Definition 5.4. Two matrices A and B are said to be **similar** if there is an invertible matrix P such that $B = P^{-1}AP$.

If A is not a diagonal matrix itself, but it is similar to a diagonal matrix D, then $A = PDP^{-1}$, for some invertible matrix P. Then

$$A^2 = (PDP^{-1})(PDP^{-1}) = PD^2P^{-1}.$$

In general, $A^n = PD^nP^{-1}$.

Compare the number of multiplications required to compute each side of this equation when n is very large, and you'll see that this formula can be quite efficient.

Eigenvalue Properties

Theorem 5.2. If A and B are similar they have the same eigenvalues. (Note: This does NOT work in the other direction.)

5.3.2 Diagonalizability

Diagonalizable

Definition 5.5. A square matrix A is said to be **diagonalizable** if $A = PDP^{-1}$ for some diagonal matrix D.

Theorem 5.3. An $n \times n$ matrix A is diagonalizable if and only if A has n linearly independent eigenvectors. In other words, A is diagonalizable if and only if the eigenvectors of A form a basis for $\mathbb{R}^n$.

If this is the case, then the columns of P are the eigenvectors of A, and the diagonal entries of D are the eigenvalues of A, in corresponding order.

Example 5.3.3. Diagnonalize $A = \begin{bmatrix} 3 & 2 & 4 \\ 2 & 0 & 2 \\ 4 & 2 & 3 \end{bmatrix}$. We did the work in example 5.2.5.

Example 5.3.4. Diagnonalize $A = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix}$. We did the work in example 5.3.1.

5.5 Complex Eigenvalues

5.5.1 Complex Arithmetic

Q: What is a complex number?

A: It is a number of the form $a + bi$ where a and b are real numbers and $i^2 = -1$.

- a is called the **real part**
- b is called the **imaginary part**.
- The **conjugate** of $a + bi$ is $a - bi$.

Properties

1. $a + b\ i = c + d\ i \quad \Leftrightarrow \quad a = c$ and $b = d$
2. $(a + b\ i) + (c + d\ i) = (a + c) + (b + d)\ i$
3. $(a + b\ i) \cdot (c + d\ i) = ac + ad\ i + bc\ i + bd\ i^2 = (ac - bd) + (ad + bc)\ i$
4. $(a + b\ i)(a - b\ i) = a^2 + b^2$

Example 5.5.1.

a. $\dfrac{i}{3 + i}$

b. $\dfrac{3 - 5i}{2 - i}$

c. $(3 - \sqrt{-4}) + (-8 + \sqrt{-25})$

d. $\dfrac{1}{3i}$

5.5.2 Complex Numbers in Equations

Example 5.5.2. Solve for x: $x^2 + 1 = 0$

Example 5.5.3. Solve the system of equations

$$\begin{aligned} x_1 \quad &+ \quad ix_2 &= 0 \\ (1-i)x_1 \quad &+ \quad (1+i)x_2 &= 0 \end{aligned}$$

Example 5.5.4. Find the eigenvalues and eigenvectors of $\begin{bmatrix} 5 & -2 \\ 1 & 3 \end{bmatrix}$

Example 5.5.5. Find the eigenvalues and eigenvectors of $\begin{bmatrix} -5 & -5 \\ 5 & -5 \end{bmatrix}$

5.5.3 Polar Form of a Complex Number

Euler's Formula

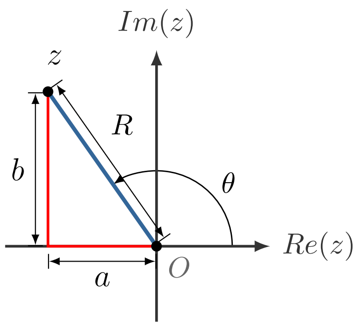
Euler's Formula

$$\begin{aligned} e^{i\theta} &= \cos\theta + i\sin\theta \\ e^{-i\theta} &= \cos(-\theta) + i\sin(-\theta) \\ &= \cos\theta - i\sin\theta \end{aligned}$$

Euler's formula allows us to plot complex numbers on the Re-Im plane. This is called an **Argand Diagram**. For example for a complex number of the form

$$z = a + bi = Re^{i\theta}$$

the Argand Diagram is shown below.



Example 5.5.6. If $C = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$, where a and b are nonzero real numbers, then the eigenvalues of C are $\lambda = a \pm bi$. Show that C can be written as a scale factor matrix $\begin{bmatrix} r & 0 \\ 0 & r \end{bmatrix}$ where $r = |\lambda|$ and a rotation matrix $\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$ where θ is the angle between the positive x-axis and the vector $[a, b]$.

Solution:

$$C = r\begin{bmatrix} a/r & -b/r \\ b/r & a/r \end{bmatrix}$$