Contents

3.1 Introduction to Determinants

2 x 2 Matrix Determinant

Definition 3.1. Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and define the determinant of A as $\det A = ad - bc$.

Definition 3.2. For any square matrix A, let A_{ij} denote the submatrix formed by deleting the i^{th} row and j^{th} column of A

Example 3.1.1. $A = \begin{bmatrix} 3 & 4 & -5 & -2 \\ 2 & -3 & 5 & -1 \\ 3 & 0 & 5 & 0 \\ 4 & 9 & 4 & 5 \end{bmatrix}$ Find A_{32}, A_{23} and A_{44}

The Determinant

Definition 3.3. For $n \geq 2$ the **determinant** of an $x \times n$ matrix $A = [a_{ij}]$ is the sum of n terms of the form $\pm a_{1j}\det A_{1j}$, with plus and minus signs alternating, where the entries $a_{11}, a_{12}, \ldots, a_{1n}$ are from the first row of A. In symbols,

$$\begin{aligned} \det A &= a_{11}\det A_{11} - a_{12}\det A_{12} + a_{13}\det A_{13} - \cdots + (-1)^{1+n} a_{1n}\det A_{1n} \\ &= \sum_{j=1}^{n} (-1)^{i+j} a_{1j}\det A_{1n} \end{aligned}$$

Example 3.1.2. For a 3×3 matrix $A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$ that would look like:

$$\det A = a\begin{vmatrix} e & f \\ h & i \end{vmatrix} - b\begin{vmatrix} d & f \\ g & i \end{vmatrix} + c\begin{vmatrix} d & e \\ g & h \end{vmatrix} = a(ei - hf) - b(di - fg) + c(dh - eg)$$

Example 3.1.3. Find the determinant of $A = \begin{bmatrix} 3 & 4 & -5 \\ 2 & -3 & 5 \\ 3 & 0 & 5 \end{bmatrix}$

Example 3.1.4. Find the determinant of $A = \begin{bmatrix} 3 & 4 & -5 & 0 \\ 2 & -3 & 5 & -1 \\ 3 & 0 & 5 & 0 \\ 4 & 9 & 0 & 5 \end{bmatrix}$

Example 3.1.5. Find the determinant of $A = \begin{bmatrix} 4 & 0 & -7 & 3 & -5 \\ 0 & 0 & 2 & 0 & 0 \\ 7 & 3 & -6 & 4 & -8 \\ 5 & 0 & 5 & 2 & -3 \\ 0 & 0 & 9 & -1 & 2 \end{bmatrix}$

Example 3.1.6. Find the determinant of the upper triangular matrix

$$A = \begin{bmatrix} a_1 & x & x & x & x \\ 0 & a_2 & x & x & x \\ 0 & 0 & \ddots & x & x \\ \vdots & \vdots & \ddots & \ddots & x \\ 0 & 0 & \cdots & 0 & a_n \end{bmatrix}$$

Cofactor Expansions

Theorem 3.1.

Given $n \times n$ matrix $A = [a_{ij}] = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$

Then the matrix A_{ij} = is the matrix formed by deleting the i^{th} row and j^{th} column of A
The (i, j)**-cofactor** of A is the number C_{ij} given by

$$C_{ij} = (-1)^{i+j} \det A_{ij}$$

then

$$\det A = a_{i1}C_{i1} + a_{i2}C_{i2} + \cdots + a_{in}C_{in}$$

This formula is called **the cofactor expansion across the i^{th} row** of A.

A similar formula can be constructed for **the cofactor expansion down the j^{th} column** of A:

$$\det A = a_{1j}C_{1j} + a_{2j}C_{2j} + \cdots + a_{nj}C_{nj}$$

3.2 Properties of Determinants

Row Operations and Determinants

Theorem 3.2. Let A be a square matrix.

1. If two rows of A are interchanged to produce B, then det $B = -\det A$.
2. If one row of A is multiplied by k to produce B, then det $B = k$ det A.
3. If a multiple of one row of A is **added** to another row to produce matrix B, then det $B =$ det A.

Example 3.2.1. Rule # 1: If you switch two rows the sign of the determinant changes.

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \qquad B = \begin{bmatrix} 3 & 4 \\ 1 & 2 \end{bmatrix}$$

Example 3.2.2. Rule # 2: If you mulitply a row of matrix A by a number, c, to make matrix B then det $B = c$ det A.

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 3 & 1 \\ 0 & 0 & -3 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1/3 \\ 0 & 0 & -3 \end{bmatrix}$$

Example 3.2.3. Rule # 3: If a multiple of one row of A is **added** to another row to produce matrix B the determinants are the same: det $B =$ det A.

$$c * R_i + R_j \longrightarrow R_j$$

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & -3 & 1 \\ 2 & 2 & -1 \end{bmatrix}$$

Other properties of determinants

Theorem 3.3. Let A be a square matrix.

1. $\det(A^{-1}) = \dfrac{1}{\det A}$.
2. $\det(AB) = (\det A)(\det B)$
3. $\det(kA) = k(\det A)$
4. $\det A^{\mathrm{T}} = \det A$

Invertible Matrices and Determinants

Theorem 3.4. A square matrix A is invertible if and only if $\det A \neq 0$.

Example 3.2.4. Find det A

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 5 & 6 & 7 \end{bmatrix}$$

Example 3.2.5. Find det A

$$A = \begin{bmatrix} 2 & 0 & 0 & 6 \\ 1 & -7 & -5 & 0 \\ 3 & 8 & 6 & 0 \\ 0 & 7 & 5 & 4 \end{bmatrix}$$

3.3 Cramer's Rule, Volume, and Linear Transformations

3.3.1 Cramer's Rule

Notation: For any $x \times n$ matrix A and any b in $\mathbb{R}^n$, let $A_i(b)$ be the matrix obtained from A by replacing column i by the vector b.

$$A_i(b) = [a_1 \ \cdots \ a_{i-1} \ \ b \ \ a_{i+1} \ \cdots \ a_n]$$

Example 3.3.1. $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$ then

$$A_1\begin{pmatrix} -5 \\ -6 \\ -7 \end{pmatrix} = \begin{bmatrix} -5 & 2 & 3 \\ -6 & 5 & 6 \\ -7 & 8 & 9 \end{bmatrix}$$

Cramer's Rule

Theorem 3.5. Let A be an invertible $x \times n$ matrix. For any b in $\mathbb{R}^n$, the unique solution x of $Ax = b$ has entries given by

$$x_i = \frac{\det\ A_i(b)}{\det\ A}$$

Example 3.3.2. Use Cramer's Rule to solve the system of equations

$$\begin{aligned} 4x_1 + x_2 &= 6 \\ 3x_1 + 2x_2 &= 7 \end{aligned}$$

$A =$ $b =$

$A_1(b) =$ $A_2(b) =$

$$x_1 = \frac{\det(A_1(b))}{\det\ A} =$$

$$x_2 = \frac{\det(A_2(b))}{\det\ A} =$$

Example 3.3.3. Use Cramer's Rule to determine the values of the parameter s for which the system has a unique solution and describe the solution.

$$\begin{aligned} s\,x_1 - 6s\,x_2 &= 3 \\ 3x_1 - 18s\,x_2 &= 5 \end{aligned}$$

3.3.2 Area and Volume

Area and Volume

Theorem 3.6.
Area: If A is a 2×2 matrix, the area of the parallelogram determined by the columns of A is $|\det A|$.

Volume: If A is a 3 matrix, the volume of the parallelepiped determined by the columns of A is $|\det A|$.

Example 3.3.4. Find the area defined by the points $A(0,-2),\ B(5,-2),\ C(-3,1),\ D(2,1)$. (ans 15)

Example 3.3.5. Find the volume of the parallelepiped with one vertex at the origin and adjacent vertices at $(1,0,-3)$, $(1,4,4)$, and $(8,2,0)$ (ans. 82)

3.3.3 Linear Transformations

Linear Transformations with Area and Volume

Theorem 3.7. Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation determined by a 3×2 matrix A. If S is the parallelogram in $\mathbb{R}^2$, then

$$\{\text{area of } T(S)\} = |\det A| \cdot \{\text{area of } S\}$$

Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be the linear transformation determined by a 3×3 matrix A. If S is the parallelepiped in $\mathbb{R}^3$, then

$$\{\text{volume of } T(S)\} = |\det A| \cdot \{\text{volume of } S\}$$

Example 3.3.6. Let S be the parallelogram determined by the vectors $b_1 = \begin{bmatrix} -3 \\ 6 \end{bmatrix}$ and $b_2 = \begin{bmatrix} -3 \\ 10 \end{bmatrix}$, and let $A = \begin{bmatrix} 6 & -3 \\ -5 & 3 \end{bmatrix}$. Compute the are of the image of S under the mapping $x \mapsto Ax$. (ans. 36)