Contents

1.1 Systems of Linear Equations . . . 2
1.2 Row Reduced and Echelon Forms . . . 5
1.2.1 Writing Solutions . . . 7
1.3 Vector Equations . . . 9
1.3.1 Algebraic Properties of Vectors in $\mathbb{R}^n$. . . . 11
1.3.2 Linear Combinations of Vectors . . . 11
1.3.3 Spanning sets . . . 12
1.4 The Matrix Equation $Ax = b$ . . . 14
1.4.1 Matrix multiplication . . . 14
1.4.2 Three ways to write the system of equations $Ax = b$ . . . 14
1.5 Solution Sets of Linear Systems . . . 18
1.5.1 Nonhomogeneous solutions . . . 19
1.7 Linear Independence . . . 20
1.8 Introduction to Linear Transformations . . . 24
1.9 The Matrix of a Linear Transformation . . . 29

1.1 Systems of Linear Equations

Linear Equation

Definition 1.1. A **Linear Equation** in variables $x_1, x_2, x_3, \ldots, x_n$ is an equation that can be written

$$a_1x_1 + a_2x_2 + a_3x_3 + \cdots + a_nx_n = b$$

where $a_1, a_2, a_3, \ldots, a_n$ and b are complex numbers.

System of Linear Equations

Definition 1.2. A **System of Linear Equations** in variables $x_1, x_2, x_3, \ldots, x_n$ is a collection of one or more linear equations that can be written

$$\left[\begin{array}{rl} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \cdots + a_{1n}x_n = & b_1 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \cdots + a_{2n}x_n = & b_2 \\ \vdots \qquad\qquad & \vdots \\ a_{m1}x_1 + a_{m2}x_2 + a_{m3}x_3 + \cdots + a_{mn}x_n = & b_m \end{array}\right]$$

where a_{ij} and b_i are complex numbers.

There are 3 possibilities for the solution to a system of equations:

1. One solution (consistent)
2. No solution (inconsistent)
3. ∞ solutions (consistent)

When we solve systems of equations there are three things that can be done:

1. Interchange two equations.
2. Multiply an equation by a nonzero constant.
3. Take a linear combination of two rows and replace either with the result.

The last item is sometimes stated as, ”Add a multiple of one equation to another and replace either with the result.”

Question: What is a ”linear combination”?

Example 1.1.1. Solve the linear system (ans. $[5, -3, 3]$)

$$\begin{cases} 2x_1 + 4x_2 + x_3 &= 1 \\ x_1 - 2x_2 - 3x_3 &= 2 \\ x_1 + x_2 - x_3 &= -1 \end{cases}$$

We don't want to have to do these calculations with all these variables so we use **Matrices**.

An $m \times n$ matrix is a rectangular array with m rows and n columns that looks like:

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & \vdots \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

Each entry a_{ij} is a complex number when working with equations but in principle can be anything. The advantage of writing the system of equations as a matrix is that we do not have to write all the variables every time. The first column only contains the coefficients of x_1, the second column only contains the coefficients of x_2 and the n^{th} column only contains the coefficients of x_n.

A system of linear equations represented as a matrix would look like:

$$\left[\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & \vdots & b_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{array}\right]$$

1.2 Row Reduced and Echelon Forms

Echelon and Row Reduced Echelon Forms

Definition 1.3. Echelon and Row Reduced Echelon Forms

Echelon Form:

1. All nonzero rows are above any row of all zeros
2. Each leading enrty of a row is in a column to the right of the leading entry of each row above it.
3. All entries in a column below a leading entry are zeros.

   2. + 3. = Leading entries are ordered strictly from left to right.

Reduced Row Echelon Form (RREF):

4. The leading entry in each row is 1.
5. Each leading entry 1 is the only nonzero entry in that column.

Important!!! Row Reduced Echelon Forms are unique

Example 1.2.1.

Echelon Form

$$\begin{bmatrix} 2 & 3 & 4 & 5 & 6 & -2 \\ 0 & 0 & 1 & 3 & -7 & 12 \\ 0 & 0 & 0 & -3 & 4 & 5 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Row Reduced Echelon Form

$$\begin{bmatrix} 1 & 3 & 0 & 0 & 6 & -2 \\ 0 & 0 & 1 & 0 & -7 & 12 \\ 0 & 0 & 0 & 1 & 4 & 5 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Pivot Positions

Definition 1.4. A **Pivot Position** in a matrix A is a location in A that corresponds to a leading 1 in the RREF form. A pivot column is a column with a pivot position.

Example 1.2.2. Are these matrices in Echelon Form?

Forward Phase

Definition 1.5. Putting a matrix in echelon form using row operations is called the **Forward Phase**.

Example 1.2.3. Row reduce to put M in Echelon Form

$$M = \begin{bmatrix} 0 & 3 & 1 & 4 & 1 & 0 \\ 2 & 6 & 4 & 0 & -2 & 2 \\ -4 & -9 & -7 & 1 & 3 & 2 \end{bmatrix}$$

1.2.1 Writing Solutions

Example 1.2.4. Write the solution for the system of equations represented by the following augmented matrices.

1. $\begin{bmatrix} 1 & 3 & 4 & 7 \\ 3 & 9 & 7 & 6 \end{bmatrix}$

2. $\begin{bmatrix} 1 & 0 & 2 & 5 \\ 0 & 1 & 5 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

3. $\begin{bmatrix} 1 & -3 & 0 & -1 & 0 & -2 \\ 0 & 1 & 0 & 0 & -4 & 1 \\ 0 & 0 & 0 & 1 & 9 & 4 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

Example 1.2.5. Mark each statement True or False

1. In some cases, a matrix may be row reduced to more that one matrix in reduced echelon form, using different sequences of row operations.
2. The row reduction algorithm applies only to augmented matrices for a linear system.
3. A basic variable in a linear system is a variable that corresponds to a pivot column in the coefficient matrix.
4. Finding a parametric description of the solution set of a linear system is the same as **solving** the system.
5. If one row in an echelon form of an augmented matrix is [0 0 0 5 0], then the associated linear system is inconsistent.
6. The echelon form of a matrix is unique.
7. The pivot positions in a matrix depend on whether row interchanges are used in the row reduction process.
8. Reducing a matrix to echelon form is called the *forward phase* of the row reduction process.
9. Whenever a system has free variables, the solution set contains many solutions.
10. A general solution of a system is an explicit description of all solutions of the system.

1.3 Vector Equations

Notation:

$\mathbb{R}$ is the real numbers

$\mathbb{R}^2$ is $\mathbb{R} \times \mathbb{R}$ the xy-plane

$\mathbb{R}^3$ is 3D space.

Vectors

Definition 1.6. A **vector** is an ordered list of numbers.... for now.

Column Vector: $v = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ \vdots \\ v_n \end{bmatrix}$ Row Vector: $v = \begin{bmatrix} v_1 & v_2 & v_3 & \cdots & v_n \end{bmatrix}$

Example 1.3.1. $\vec{w}_1 = \begin{bmatrix} 1 \\ -5 \end{bmatrix}$ is a vector in $\mathbb{R}^2$.

$\vec{w}_2 = (1, -5) =< 1, -5 >$ is a vector in $\mathbb{R}^2$

$\vec{w}_3 = \begin{bmatrix} 1 & -5 \end{bmatrix}$ is a vector in $\mathbb{R}^2$

What is the difference between $\vec{w}_1$, $\vec{w}_2$, and $\vec{w}_3$?

A **scalar multiple** of vector $\vec{v}$ is the vector $c\vec{v}$ obtained by multiplying every element in vector $\vec{v}$ by scalar c. For example

$$2 * \vec{w}_1 = 2 * \begin{bmatrix} 1 \\ -5 \end{bmatrix} = \begin{bmatrix} 2 \\ -10 \end{bmatrix}$$

Adding vectors is done by adding the corresponding coordinates. For example

$$\begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ \vdots \\ v_n \end{bmatrix} + \begin{bmatrix} w_1 \\ w_2 \\ w_3 \\ \vdots \\ w_n \end{bmatrix} = \begin{bmatrix} v_1 + w_1 \\ v_2 + w_2 \\ v_3 + w_3 \\ \vdots \\ v_n + w_n \end{bmatrix}$$

Example 1.3.2. Vectors can be represented as arrows in the plane. Graph $\vec{w} = \begin{bmatrix} 1 \\ -5 \end{bmatrix}$ and $\vec{v} = \begin{bmatrix} -3 \\ -2 \end{bmatrix}$ and $\vec{w} + \vec{v}$. (Parallelogram rule)

Example 1.3.3. Draw the following vectors $\vec{u} = \begin{bmatrix} 1 \\ -3 \end{bmatrix}$, $2\vec{u}$, and $-2\vec{u}$

1.3.1 Algebraic Properties of Vectors in $\mathbb{R}^n$.

Algebraic Properties of Vectors

For all u, v, and w in $\mathbb{R}^n$ and all scalars a and b:

1. $u + v = v + u$
2. $(u + v) + w = u + (v + w)$
3. $u + 0 = 0 + u = u$
4. $u + (-u) = -u + u = u - u = 0$
5. $\mathrm{a}(u + v) = \mathrm{a}u + \mathrm{a}v$
6. $(\mathrm{a} + \mathrm{b})u = \mathrm{a}u + \mathrm{b}u$
7. $(\mathrm{ab})u = \mathrm{a}(\mathrm{b}u)$
8. $1u = u$

1.3.2 Linear Combinations of Vectors

Linear Combination

Definition 1.7. Given vectors $v_1, v_2, v_3, \ldots, v_n$ and constants $c_1, c_2, c_3, \ldots, c_n$ the vector

$$y = c_1 v_1 + c_2 v_2 + c_3 v_3 + \cdots + c_n v_n$$

is called a **Linear Combinations** of the vectors v_i with weights c_i.

Example 1.3.4. Given $\vec{v}_1 = \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix} -3 \\ 1 \\ 8 \end{bmatrix}$, $y_1 = \begin{bmatrix} 1 \\ -5 \\ -3 \end{bmatrix}$, and $y_2 = \begin{bmatrix} -\frac{7}{2} \\ -5 \\ -3 \end{bmatrix}$, are y_1 or y_2 a linear combination of $\vec{v}_1$ and $\vec{v}_2$.

Solution: If y_1 is a linear combination of $\vec{v}_1$ and $\vec{v}_2$ then y_1 must solve the following equation for some values of x_1 and x_2:

$$y_1 = x_1 \vec{v}_1 + x_2 \vec{v}_2$$

$$\begin{bmatrix} 1 \\ -5 \\ -3 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix} + x_2 \begin{bmatrix} -3 \\ 1 \\ 8 \end{bmatrix}$$

That is a system of equations and we can solve it using an augmented matrix and row reduction:

$$\left[\begin{array}{cc|c} 1 & -3 & 1 \\ 0 & 1 & -5 \\ -2 & 8 & -3 \end{array}\right]$$

1.3.3 Spanning sets

Spanning Set

Definition 1.8. If $v_1, v_2, v_3, \dots, v_p$ are vectors in $\mathbb{R}^n$ then the set of all linear combinations of $v_1, v_2, v_3, \dots, v_p$ is called the **Span of** $v_1, v_2, v_3, \dots, v_p$ represented $\text{span}\{v_1, v_2, v_3, \dots, v_p\}$ and is the subset of $\mathbb{R}^n$ spanned by $v_1, v_2, v_3, \dots, v_p$.

$$\text{span}\{v_1, v_2, v_3, \dots, v_p\} = c_1v_1 + c_2v_2 + c_3v_3 + \cdots + c_pv_p$$

The previous example could have been stated as: Is y_1 or y_2 in $\text{span}\{v_1, v_2\}$

Example 1.3.5. Determine if $b = \begin{bmatrix} 11 \\ -5 \\ 9 \end{bmatrix}$ is in the span of the column vectors that form

$$A = \begin{bmatrix} 1 & -2 & -6 \\ 0 & 3 & 7 \\ 1 & -2 & 5 \end{bmatrix} = [v_1 \quad v_2 \quad v_3]$$

1.4 The Matrix Equation $Ax = b$

1.4.1 Matrix multiplication

Matrix Multiplication with a Vector

Definition 1.9. If A is the matrix with columns $a_1, a_2, a_3, \ldots, a_n$ and x is in $\mathbb{R}^n$ the the **product** Ax is defined as

$$Ax = \begin{bmatrix} a_1 & a_2 & \cdots & a_n \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = a_1x_1 + a_2x_2 + \cdots + a_nx_n$$

Example 1.4.1. Multiply $\begin{bmatrix} -4 & 2 \\ 1 & 6 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 6 \end{bmatrix}$

Example 1.4.2. Find Ax given that $A = \begin{bmatrix} 1 & -3 & 5 \\ 0 & 2 & -3 \end{bmatrix}$ and $x = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$

1.4.2 Three ways to write the system of equations $Ax = b$

1. Write $Ax = b$ explicitly in the the form

$$\begin{bmatrix} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \cdots + a_{1n}x_n = & b_1 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \cdots + a_{2n}x_n = & b_2 \\ \vdots & \vdots \\ a_{m1}x_1 + a_{m2}x_2 + a_{m3}x_3 + \cdots + a_{mn}x_n = & b_m \end{bmatrix}$$

2. Write as a vector equation (Linear combination of column vectors)

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n = b$$

where $a_i = \begin{bmatrix} a_{1i} \\ a_{2i} \\ \vdots \\ a_{mi} \end{bmatrix}$

3. Write as an augmented matrix

$$\left[\begin{array}{cccc|c} a_1 & a_2 & \cdots & a_n & b \end{array}\right]$$

Theorem about linear combinations

Theorem 1.1. Let A be an $m \times n$ matrix and b be a column vector in $\mathbb{R}^m$. Then the following statements are logically equivalent. That is, for a particular A, either they are all true statements or all false.

1. The equation $Ax = b$ has a solution
2. b is a linear combination of the columns of A
3. b is in span$\{a_1, a_2, \ldots, a_n\}$
4. $Ax = b$ is consistent.

Theorem about spanning $\mathbb{R}^n$

Theorem 1.2. Let A be an $m \times n$ matrix. Then the following statements are logically equivalent. That is, for a particular A, either they are all true statements or all false.

1. For each b in $\mathbb{R}^m$, The equation $Ax = b$ has a solution
2. Each b in $\mathbb{R}^m$ is a linear combination of the columns of A
3. The columns of A span $\mathbb{R}^m$.
4. A has a pivot position in every row.

Example 1.4.3. Write as a vector equation and as matrix equation
$$\begin{aligned} x_1 - x_3 &= 5 \\ -2x_1 + x_2 + 2x_3 &= -6 \\ 2x_2 + 2x_3 &= -4 \end{aligned}$$

Example 1.4.4. Let $\vec{u} = \begin{bmatrix} 0 \\ 4 \\ 4 \end{bmatrix}$ and $A = \begin{bmatrix} 3 & -5 \\ -2 & 6 \\ 1 & 1 \end{bmatrix}$. Is $\vec{u}$ spanned by the columns of A?

Example 1.4.5. Given $A = \begin{bmatrix} 1 & -3 & -4 \\ -3 & 2 & 6 \\ 5 & -1 & -8 \end{bmatrix}$ answer the following.

1. How many rows of A contain a pivot position?
2. Does the equation $Ax = b$ have a solution for each $b \in \mathbb{R}^3$?
3. Can each vector in $\mathbb{R}^3$ be written as a linear combination of the columns of matrix A?
4. Do the columns of A span $\mathbb{R}^3$?

Example 1.4.6. Construct a 3×3 matrix, not in echelon form, whose columns do NOT span $\mathbb{R}^3$. Show that your matrix has the desired property.

Example 1.4.7. Find a column of the matrix $A = \begin{bmatrix} 12 & -7 & 11 & -9 & 5 \\ -9 & 4 & -8 & 7 & -3 \\ -6 & 11 & -7 & 3 & -9 \\ 4 & -6 & 10 & -5 & 12 \end{bmatrix}$ that can be deleted and yet have the remaining matrix columns span $\mathbb{R}^4$.

1.5 Solution Sets of Linear Systems

Homogeneous Equations

Definition 1.10. A system of linear equations is said to be **homogeneous** if it can be written in the form $Ax = 0$. The **trivial solution** is the solution $x = 0$.

Theorem 1.3. The trivial solution is ALWAYS as solution to the homogeneous equation $Ax = 0$. The homogeneous equation $Ax = 0$ has a nontrivial solution if and only if the equation has at least one free variable.

Example 1.5.1. $\begin{bmatrix} 1 & 0 & 4 \\ -3 & 1 & -6 \\ 0 & 2 & 12 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

Example 1.5.2. Write the general solution of $x_1 + 9x_2 - 4x_3 = 7$

1.5.1 Nonhomogeneous solutions

Example 1.5.3. Describe all solutions of $Ax = b$ where

$$A = \begin{bmatrix} 1 & 0 & 4 \\ -3 & 1 & -6 \\ 0 & 2 & 12 \end{bmatrix} \qquad \text{and} \qquad b = \begin{bmatrix} 4 \\ -3 \\ 18 \end{bmatrix}$$

Particular and homogeneous solutions theorem

Theorem 1.4. Suppose the equation $Ax = b$ is consistent for some given b, and let p be a solution. Then the solution set of $Ax = b$ is the set of all vectors of the from $w = p + v_h$ where v_h is any solution of the homogeneous equation $Ax = 0$.

1.7 Linear Independence

Linear Independence

Definition 1.11. A set of vectors $\{v_1, v_2, \ldots, v_n\}$ in $\mathbb{R}^n$ is said to be **Linearly Dependent** if there exists a set of constants $c_1, c_2, \ldots, c_n$, not all zero, such that

$$c_1 v_1 + c_2 v_2 + \cdots + c_n v_n = 0.$$

The set is **Linearly Independent** otherwise.

Example 1.7.1. Are the vectors $v_1 = \begin{bmatrix} 1 \\ -3 \\ 0 \end{bmatrix}$, $v_2 = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}$, $v_3 = \begin{bmatrix} 4 \\ -6 \\ 12 \end{bmatrix}$ linearly independent? If possible find a dependence relation among them.

Sometimes we talk about the linear independence of the matrix columns.

Nontrivial solutions and Dependence

Theorem 1.5. The columns of matrix $A = [a_1 \;\; a_2 \;\; \cdots \;\; a_n]$ are linearly dependent if $Ax = 0$ has a nontrivial solution.

Example 1.7.2. Determine if the columns of matrix $A = \begin{bmatrix} 0 & 0 & -3 \\ 0 & 5 & 4 \\ 2 & -8 & 1 \end{bmatrix}$ are linearly dependent.

Some characterizations of linearly dependent sets

Theorem 1.6. Some characterizations of linearly dependent sets:

1. One vector is always independent
2. Two vectors are dependent if one is a multiple of the other.
3. A set of vectors $\{v_1, v_2, \ldots, v_n\}$ is linearly dependent if at least one can be written as a multiple of the others.

$$v_i = c_1 v_1 + \cdots + c_{i-1} v_{i-1} + c_{i+1} v_{i+1} + \cdots + c_n v_n.$$

Example 1.7.3. Let $u = \begin{bmatrix} 3 \\ 2 \\ -4 \end{bmatrix}$, $v = \begin{bmatrix} -6 \\ 1 \\ 7 \end{bmatrix}$, $w = \begin{bmatrix} 0 \\ -5 \\ 2 \end{bmatrix}$, $z = \begin{bmatrix} 3 \\ 7 \\ -5 \end{bmatrix}$.

Is w a linear combination of u, v, and z?

Is the set $\{u, v, w, z\}$ linearly independent?

More characterizations of linearly dependent sets

Theorem 1.7. If a set contains more vectors than there are entries in each vector then the set is linearly dependent.

Theorem 1.8. If a set $S = \{v_1, v_2, \ldots, v_n\}$ contains the zero vector it is linearly dependent.

Example 1.7.4. Each statement is either true (in all cases) or false (for at least one example). If false, construct a specific example, called a counterexample, to show that the statement is not always true. If true, give a justification, not just a specific example

1. The columns of a matrix A are linearly independent if the equation $Ax = 0$ has the trivial solution.

2. If S is a linearly dependent set, then each vector is a linear combination of the other vectors in S.

3. The columns of a 4 x 5 matrix are linearly dependent.

4. If x and y are linearly independent and if $\{x, y, z\}$ is linearly dependent, then z is in $\text{Span}\{x, y\}$.

5. If v_1 and v_2 are in $\mathbb{R}^4$ and v_2 is not a scalar multiple of v_1, then $\{v_1, v_2\}$ is linearly independent.

6. If $v_1, \ldots, v_4$ are in $\mathbb{R}^4$ and v_3 is **not** a linear combination of $v_1,\ v_2,\ v_4$, then $\{v_1, v_2, v_3, v_4\}$ is linearly independent.

7. If $v_1, \ldots, v_4$ are linearly independent vectors in $\mathbb{R}^4$, then $\{v_1, v_2, v_3\}$ is also linearly independent.

Example 1.7.5. Determine by inspection if the given set is linearly independent.

1. $\left\{ \begin{bmatrix} 5 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 8 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \end{bmatrix} \right\}$

2. $\left\{ \begin{bmatrix} 4 \\ -2 \\ 6 \end{bmatrix}, \begin{bmatrix} 6 \\ -3 \\ 9 \end{bmatrix} \right\}$

3. $\left\{ \begin{bmatrix} 3 \\ 5 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 6 \\ 5 \\ 4 \end{bmatrix} \right\}$

Example 1.7.6. Justify your answers.

1. For what values of h is v_3 in Span$\{v_1, v_2\}$?

2. For what values of h is $\{v_1, v_2, v_3\}$ linearly dependent?

$$v_1 = \begin{bmatrix} 1 \\ -5 \\ -3 \end{bmatrix}, \qquad v_2 = \begin{bmatrix} -2 \\ 10 \\ 6 \end{bmatrix}, \qquad v_3 = \begin{bmatrix} 2 \\ -9 \\ h \end{bmatrix}$$

1.8 Introduction to Linear Transformations

$Ax = b$ is a matrix equation. We can also think of the matrix A as doing something to the vector x. We say that A "acts" on x by multiplication. This produces a new vector Ax.

Transformations and domains

Definition 1.12. A **Transformation** (or **Function** or **Mapping**) T from $\mathbb{R}^n$ to $\mathbb{R}^m$ is a rule that assigns to each vector x in $\mathbb{R}^n$ a vector $T(x)$ in $\mathbb{R}^m$. The set $\mathbb{R}^n$ is called the **Domain** of T, and $\mathbb{R}^m$ is called the **Codomain** of T. The notation

$$T : \mathbb{R}^n \to \mathbb{R}^m$$

indicates that the domain of T is $\mathbb{R}^n$ and the codomain is $\mathbb{R}^m$.

For x in $\mathbb{R}^n$, the vector $T(x)$ in $\mathbb{R}^m$ is called the **image** of x. The set of all images is the **Range** of T.
The transformation $T(x) = Ax$ is sometimes written as $x \longmapsto Ax$ and

$$\text{the domain is} \quad \mathbb{R}^n$$

$$\text{the codomain is} \quad \mathbb{R}^m$$

$$\text{the range is} \quad \{T(x) : x \in \mathbb{R}^n\}$$

Example 1.8.1. Given $A = \begin{bmatrix} 1 & 1 \\ 2 & -1 \end{bmatrix}$, $x_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $x_2 = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$, $x_3 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$.

1. Plot the triangle with vertices x_1, x_2, x_3.
2. Plot the triangle with vertices Ax_1, Ax_2, Ax_3.

Linear Transformations

Definition 1.13. A transformation T is **Linear** if

1. $T(\mathbf{u}+\mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})$ for all $\mathbf{u}$ and $\mathbf{v}$ in the domain of T:
2. $T(c\mathbf{u}) = cT(\mathbf{u})$ for all scalars c and all $\mathbf{u}$ in the domain of T.

$T(0) =$

$T(c\mathbf{u} + d\mathbf{v}) =$

Example 1.8.2. Define $T(x) = Ax$ where $A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & -1 & 2 & 3 \end{bmatrix}$. Find the transformation $x \longmapsto Ax$ of $x = \begin{bmatrix} 0 \\ 1 \\ 2 \\ 3 \end{bmatrix}$. What are the domain and codomain of A.

Example 1.8.3. Define the transformation $T(x) = Ax$ where $A = \begin{bmatrix} 1 & 0 \\ -3 & 1 \end{bmatrix}$. Find the transformation of the square in the first quadrant with one vertex at $(0, 0)$ and side length 2. Sketch the square and transformation of the square.

Example 1.8.4. Find all the vectors that map onto $\begin{bmatrix} -2 \\ -2 \end{bmatrix}$ given the matrix transformation defined by $A = \begin{bmatrix} 1 & -5 & -7 \\ -3 & 7 & 5 \end{bmatrix}$. Need to solve $Ax = \begin{bmatrix} -2 \\ -2 \end{bmatrix}$

Example 1.8.5. Given

$$A = \begin{bmatrix} 1 & 3 & 9 & 2 \\ 1 & 0 & 3 & -4 \\ 0 & 1 & 2 & 3 \\ -2 & 3 & 0 & 5 \end{bmatrix}, \qquad b = \begin{bmatrix} -1 \\ 3 \\ -1 \\ 4 \end{bmatrix}$$

1. Find all x in $\mathbb{R}^4$ that are mapped into the zero vector by the transformation $x \longmapsto Ax$.

2. Is b in the range of the linear transformation $x \longmapsto Ax$? Why or why not?

Example 1.8.6. A transformation T is linear if $T(c\mathbf{u}) = cT(\mathbf{u})$ for all scalars c and all $\mathbf{u}$ is the domain of T. Show that the transformation T defined by $T(x_1, x_2) = (4x_1 - 2x_2, 3|x_2|)$ is not linear.

Example 1.8.7. If a transformation satisfies $T(c\mathbf{u} + d\mathbf{v}) = cT(\mathbf{u}) + dT(\mathbf{v})$ for all vectors $\mathbf{u}$, $\mathbf{v}$ in the domain of T and all scalars c, d, it must be linear. Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be the transformation that reflects each vector (x_1, x_2, x_3) through the plane $x_3 = 0$ onto $T(x) = (x_1, x_2, -x_3)$. Show that T is a linear transformation.

1.9 The Matrix of a Linear Transformation

Identity Matrix and Basis

Definition 1.14. The **identity matrix** I_n in the $n \times n$ matrix with 1's on the main diagonal and zeros everywhere else.

$I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

A **basis** is a minimal spanning set. (a more detailed definition shows up later).

The **Standard Basis** for $\mathbb{R}^n$ is the set of vectors $e_i = \begin{bmatrix} 0 & \cdots & 0 & 1 & 0 & \cdots & 0 \end{bmatrix}$ where the 1 is in the i^{th} position.
If you know what a transformation does to the basis elements you know what it does to all vectors.

$$T(c_1\mathbf{e_1} + c_2\mathbf{e_2} + \cdots + c_n\mathbf{e_n}) = c_1T(\mathbf{e_1}) + c_2T(\mathbf{e_2}) + \cdots + c_nT(\mathbf{e_n})$$

Example 1.9.1. For $\mathbb{R}^2$ the standard basis is $B = \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}$. If $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in \mathbb{R}^2$ then $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} x_1 + \begin{pmatrix} 0 \\ 1 \end{pmatrix} x_2$. Use the definition to write $T\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$
What is the basis for $\mathbb{R}^3$?

Example 1.9.2. Suppose $T : \mathbb{R}^2 \to \mathbb{R}^4$ and we know $T(e_1) = \begin{bmatrix} 3 \\ 4 \\ 5 \\ 1 \end{bmatrix}$ and $T(e_2) = \begin{bmatrix} -3 \\ 2 \\ 5 \\ -7 \end{bmatrix}$. Find a matrix A that has the same mapping.

Example 1.9.3. Assume that T is a linear transformation. Find the standard matrix of T where T first performs a horizontal shear that transforms e_2 into $e_2 + 12e_1$ (leaving e_1 unchanged) and then reflects points through the line $x_2 = -x_1$.

One-to-one and Onto mappings

Definition 1.15. A mapping $T : \mathbb{R}^n \to \mathbb{R}^m$ is said to be **onto** $\mathbb{R}^m$ if each b in $\mathbb{R}^m$ is the image of at least one x in $\mathbb{R}^n$

A **one-to-one transformation** is a transformation where each x in the domain is mapped to exactly one element in the range. In other words, $x \longmapsto Ax$ is a unique map.

Example 1.9.4. $A = \begin{bmatrix} 1 & -5 & -7 \\ -3 & 7 & 5 \end{bmatrix}$ is not a one-to-one map because the vector $x = \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix} + \begin{bmatrix} -3 \\ -2 \\ 1 \end{bmatrix} t$ maps to $\begin{bmatrix} -2 \\ -2 \end{bmatrix}$. (see **Example 1.8.4**)

one-to-one and onto theorems

Theorem 1.9. Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation, and let A be the standard matrix for T. Then:

1. T is one-to-one if and only if the equation $T(x) = 0$ has only the trivial solution.
2. T is one-to-one if and only if the equation $Ax = 0$ has only the trivial solution.
3. T maps $\mathbb{R}^n$ onto $\mathbb{R}^m$ if and only if the columns of A span $\mathbb{R}^m$.
4. T is one-to-one if and only if the columns of A are linearly independent.

Example 1.9.5. $T(x) : \mathbb{R}^3 \to \mathbb{R}^2$ is defined by the matrix $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$. Show that the transformation is **onto** $\mathbb{R}^2$. (Hint: Need to show that you can get to any vector in $\mathbb{R}^2$ (what does that look like?) from some vector in $\mathbb{R}^3$ (what does that look like?)).

Example 1.9.6. $T(x) : \mathbb{R}^3 \to \mathbb{R}^2$ is defined by the matrix $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$. Show that the transformation is **NOT** one-to-one. (Hint: Need to show that you can get to some vector in $\mathbb{R}^2$ from multiple vectors in $\mathbb{R}^3$ (what does that look like?)).

Example 1.9.7. Show that T is a linear transformation by finding a matrix that implements the mappings

1. $T(x_1, x_2, x_3, x_4) = (x_1 + 8x_2, 0, x_1 - 5x_2 + 6x_4, x_2 - 3x_3)$
2. $T(x_1, x_2, x_3) = (x_1 - 5x_2 + 6x_3, x_2 - 3x_3)$