Math 176 Calculus – Sec. 8.3: Moments and Centers of Mass

I. Finite Masses
   A. One Dimensional Cases
      1. Introduction
         Recall the difference between Mass and Weight.
         a. Mass is the amount of “stuff” (matter) that makes up an object.
         b. Weight is a measure of force that results from gravity acting on a mass.
         c. Newton’s Second Law: \( F = m \cdot g \), where \( m \) = amount of mass, \( g \) is the acceleration of gravity.

         In general a force \( F \), is equal to the product of a mass and acceleration, i.e. \( F = m \cdot a \).

         | System of Measurement | Measure of Mass | Measure of Force |
         |------------------------|-----------------|-----------------|
         | U.S.                   | slug            | pound = (slug)/(ft/sec\(^2\)) |
         | International          | kilogram        | newton = (kilogram)/(m/sec\(^2\)) |
         | C-G-S                  | gram            | dyne = (gram)/(cm/sec\(^2\)) |

         Conversions:
         1 pound = 4.448 newtons
         1 slug = 14.59 kilograms
         1 newton = 0.2248 pounds
         1 kilogram = 0.06854 slugs
         1 dyne = 0.002248 pounds
         1 gram = 0.00006854 slugs

      2. Finite Masses Along a Line.
         a. We begin by considering an idealized situation in which a mass \( m \) is concentrated at a point. If \( x \) is the directed distance between this point-mass and another point \( P \), then the moment of \( m \) about the point \( P \) is given by moment = \( m \otimes x \), where \( x \) is the length of the moment arm.

         Now let’s imagine masses \( m_1, m_2, m_3 \) on a rigid x-axis supported by a fulcrum at the origin.

         \[
         \begin{array}{c}
         m_1 \\
         0 \\
         m_2 \\
         \Delta \\
         m_3 \\
         \end{array}
         \]

         fulcrum

         Each mass \( m_k \) exerts a downward force (\( m_k \cdot g \)). Each of these masses has a tendency to turn the axis about the origin, the way you turn a seesaw. This turning effect is called torque. Torque \( T = (m_k \cdot g)(x_k) \) where \( x_k \) is the signed distance from the point of application to the origin. From observations we know that masses to the left of the origin exert negative (counterclockwise) torque, while masses to the right of the origin exert positive (clockwise) torque. (Note, in general Torque, \( T = F \cdot L \), where \( F \) is a force and \( L \) is the distance from where
the force is applied to the center or origin. Think about turning a wrench to loosen a bolt, the longer the handle of the wrench the more torque you can exert to loosen the bolt (assuming you apply the force at the end of the handle.)

The sum of all the torques measures the tendency of a system to rotate about the origin. This is called the **system torque**. (System Torque = \( m_1gx_1 + m_2gx_2 + m_3gx_3 + \ldots + m_ngx_n \)) The system will balance on the fulcrum if and only if the system torque = 0 (we say such a system is in **equilibrium/balance**.)

Let’s examine the system torque formula a little more: \( T = m_1gx_1 + m_2gx_2 + m_3gx_3 + \ldots + m_ngx_n \), since each \( m_kx_k \) is a moment, we call there sum the **moment of the system about the origin, \( M_o \)**. (Moment, \( Mo = \text{product of mass } m \text{ of a particle by its directed distance from that pt } x \) ) \( M_o = \sum_{k=1}^{n} m_kx_k \), where \( n \) is the number of masses. Notice that a system will be in equilibrium/balance if and only if \( M_o = 0 \), since \( g \) is rarely zero.

We usually can’t move the masses in a system, but we can move the Fulcrum in order to make the system balance, i.e so that \( M_o = 0 \), (\( T = 0 \)). We label that point \( \bar{x} \).

The system’s center of mass, \( \bar{x} \), is given by the formula:

\[
\bar{x} = \frac{\sum_{k=1}^{n} m_kx_k}{\sum_{k=1}^{n} m_k} = \frac{M_o}{M} = \frac{\text{system moment about origin}}{\text{system mass}}
\]

\[ m_1 \quad 0 \quad m_2 \quad \bar{x} \quad m_3 \]

\[ \leftarrow \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \rightarrow \]

\[ x_1 \quad \Delta \quad x_2 \quad :. \quad x_3 \]

\[ \uparrow \quad \uparrow \quad \downarrow \quad \text{fulcrum} \quad \text{center of mass} \]


**Moment about the origin** : \( M_o = \sum_{k=1}^{n} m_kx_k \)

**Mass** : \( M = \sum_{k=1}^{n} m_k \)

**Center of Mass** : \( \bar{x} = \frac{M_o}{M} = \frac{\sum_{k=1}^{n} m_kx_k}{\sum_{k=1}^{n} m_k} \)
c. Examples
1.) Three bodies of mass 6kg, 4kg & 10kg are located at $x_1 = -2$, $x_2 = 4$ & $x_3 = 9$ respectively. If the distances are measured in meters, find the center of mass.

2.) Two children want to balance on a seesaw. One is 35lbs and sits 4 ft from the center. Where should the other child sit if he weighs 50lbs?

B. Two Dimensional Cases

1. Masses Distributed Over a Plane Region
   a. Development of formulas
      1.) Suppose we have a finite collection of masses in a plane.
      2.) Each mass $m_k$ is located at the pt $(x_k, y_k)$.
      3.) The system mass: $M = \sum_{k=1}^{n} m_k$
      4.) Each mass has a moment about each axis (the x-axis and the y-axis).
         a.) $m_k$’s moment about the x-axis is given by $m_k y_k$.
         b.) $m_k$’s moment about the y-axis is given by $m_k x_k$.
      5.) The system's entire moments about the two axes will be:
         a.) Moment about the x-axis: $M_x = \sum_{k=1}^{n} m_k x_k$
         b.) Moment about the y-axis: $M_y = \sum_{k=1}^{n} m_k y_k$
      6.) The system’s Center of Mass is defined to be $(\bar{X}, \bar{Y})$ where
         $$\bar{X} = \frac{M_y}{M} = \frac{\sum_{k=1}^{n} (m_k x_k)}{\sum_{k=1}^{n} m_k} \quad \text{and} \quad \bar{Y} = \frac{M_x}{M} = \frac{\sum_{k=1}^{n} (m_k y_k)}{\sum_{k=1}^{n} m_k}$$
      7.) Note: with $\bar{X}$ and $\bar{Y}$ defined this way:
         a.) The system will balance about the line $x = \bar{X}$.
         b.) The system will balance about the line $y = \bar{Y}$. 
2. Examples
   a. Three particles having masses 2, 5, 8 units are located at (-1,3), (0,-7) & (2,1), respectively. Find \( (\bar{X}, \bar{Y}) \).

b. Find the centroid (center of mass with uniform density) of the region shown, not by integration, but by locating the centers of the rectangles and triangles and treating them as point masses.

1.)

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NOTE: Although the following examples are not truly finite masses in a plane, we will find the centroid using the addition of the point masses in each region.
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2.)
3.)

Note: The centroid of a triangle is located at the pt of intersection of the medians. A median is the line from a vertex to the midpt of the opposite side. Medians intersect at a pt \( \frac{2}{3} \) of the way from each vertex (along the median) to the opposite side.

4.)
II. Uniform Density Along a Line or Over a Plane

A. Continuous Mass Distribution Along a Line - Wires and Thin Rods with Varying Density

1. **Recall:** Density, \( \delta \), is defined to be a material's mass per unit volume. However, we often use different units for our own convenience, (i.e. mass per unit length or mass per unit area.).

2. For the center of mass of a rod or thin strip of metal, we can model the distribution of mass with a **continuous function**, then the summation formulas we've seen thus far can be replaced with integrals.

3. **Derivation of Formula:** Take a long, thin strip of metal lying along the x-axis from \( x = a \) to \( x = b \). Partition the strip into small pieces of mass \( \Delta m_k \). The kth piece is \( \Delta x_k \) units long and lies approximately \( x_k \) units from the origin.

   \[
   \Delta m_k \quad \leftarrow \quad \underbrace{\ldots \ldots \ldots \ldots \ldots}_{a \quad x_k \quad b}
   \]

   **Observe:**
   1.) The strip’s center of mass is nearly the same as that of the system of point masses we would get by attaching each mass \( \Delta m_k \) to the point \( x_k \); so we have
   \[
   -x \approx \frac{\text{system moment about origin}}{\text{system mass}}
   \]

   2.) The moment of each piece of the strip about the origin is approximately \( x_k \Delta m_k \), so the system moment is approximately: \( M_o = \sum_{k=1}^{n} x_k \Delta m_k \)

3.) If the density of the strip at \( x_k \) is \( \delta(x_k) \), expressed in terms of mass/unit length, and \( \delta \) is continuous, then: \( \Delta m_k = \rho(x_k) \Delta x_k \)

   Therefore:
   \[
   x \approx \frac{\text{system moment about origin}}{\text{system mass}} \approx \frac{\sum_{k=1}^{n} x_k \Delta m_k}{\sum_{k=1}^{n} \Delta m_k} = \frac{\sum_{k=1}^{n} x_k \rho(x_k) \Delta x_k}{\sum_{k=1}^{n} \rho(x_k) \Delta x_k}
   \]

4.) Notice that the numerator and denominator of the last sum are both Riemann Sums. The numerator is a Riemann sum for the continuous function \( x \delta(x) \) and the denominator is a Riemann sum for the function \( \delta(x) \), both over the closed interval \([a,b]\). Thus as \( |P| \to 0 \) we arrive at the following formula:

   \[
   -x = \frac{\text{system moment about origin}}{\text{system mass}} = \frac{M_o}{M} = \frac{\int_{a}^{b} x \rho(x) \, dx}{\int_{a}^{b} \rho(x) \, dx}
   \]
4. Formulas for Moment, Mass & Center of Mass of a Thin Rod Along the x-axis with Density Function, \( \rho(x) \):

\[
\text{Moment about the origin: } M_o = \int_{a}^{b} [x \rho(x)] \, dx
\]

\[
\text{Mass: } M = \int_{a}^{b} [\rho(x)] \, dx
\]

\[
\text{Center of Mass: } \bar{X} = \frac{M_o}{M} = \frac{\int_{a}^{b} [x \rho(x)] \, dx}{\int_{a}^{b} [\rho(x)] \, dx}
\]

5. Comments:
1.) The center of mass of a straight, thin rod or strip of constant density lies halfway between its ends.

2.) We can treat a rod of variable thickness as a rod of variable density.

6. Example
1.) A 16cm long wire has a linear density measured in g/cm, given by \( \rho(x) = \sqrt{x} \), \( 0 < x \leq 16 \). Find the center of mass.

B. Thin, Flat Planes

1. Development of formulas
a. We assume that the mass is continuously distributed in one of these plates. Often such a plate is called a planar lamina. Then just like when we moved from the point mass system in 1-D, to the thin strips, the formulas given above involving summations will become formulas involving integrals.

b. Imagine cutting a planar lamina in the xy-plane into thin strips parallel to one of the axes. The center of mass of a sample strip is \( (\tilde{x}, \tilde{y}) \).

c. Each sample strip's mass \( \Delta m \) is treated as if it were concentrated at \( (\tilde{x}, \tilde{y}) \).

d. The moment of each sample strip about\( \begin{align*}
\text{the \ y-axis \ is \ } & \tilde{x} \Delta m \\
\text{the \ x-axis \ is \ } & \tilde{y} \Delta m
\end{align*} \).
e. The system's center of mass becomes

\[
\overline{X} = \frac{M_y}{M} = \frac{\sum_{k=1}^{n} (\bar{x}_k \Delta m_k)}{\sum_{k=1}^{n} \Delta m_k}, \quad \text{and} \quad \overline{Y} = \frac{M_x}{M} = \frac{\sum_{k=1}^{n} (\bar{y}_k \Delta m_k)}{\sum_{k=1}^{n} \Delta m_k}
\]

g. Taking the limits as \( \Delta m_k \to 0 \) we get the center of mass formulas:

\[
\overline{X} = \frac{M_y}{M} = \frac{\int \bar{x} \, dm}{\int dm} \quad \text{and} \quad \overline{Y} = \frac{M_x}{M} = \frac{\int \bar{y} \, dm}{\int dm}
\]

2. Formulas for Moment, Mass & Center of Mass of a Lamina with Density Function, \( \rho(x) \):

**Mass:** \( M = \int_{a}^{b} dm = \int_{a}^{b} \rho(x) \, dA \)

**Moment about the \( y \)-axis:** \( M_y = \int_{a}^{b} \bar{x} \, dm = \int_{a}^{b} \bar{x} \rho(x) \, dA \)

**Moment about the \( x \)-axis:** \( M_x = \int_{a}^{b} \bar{y} \, dm = \int_{a}^{b} \bar{y} \rho(x) \, dA \)

**Center of Mass:** \( \overline{X} = \frac{M_y}{M}, \quad \overline{Y} = \frac{M_x}{M} \)

When integrating wrt \( x \): \( \bar{x} = x \), \( \bar{y} = \frac{f(x) + g(x)}{2} \) and \( dA = \left[ f(x) - g(x) \right] \, dx \), where \( f(x) = \text{top curve} \) and \( g(x) = \text{bottom curve} \) and \( \rho(x) = \rho \) because it is a constant for our examples. Therefore, the formulas above can be rewritten as

**Mass:** \( M = \int_{a}^{b} dm = \int_{a}^{b} \rho \left( f(x) - g(x) \right) dx \)

**Moment about the \( y \)-axis:** \( M_y = \int_{a}^{b} \rho x \left( f(x) - g(x) \right) dx \)

**Moment about the \( x \)-axis:** \( M_x = \int_{a}^{b} \rho \left[ \frac{f(x) + g(x)}{2} \right] \left( f(x) - g(x) \right) dx \)

\( = \frac{1}{2} \int_{a}^{b} \rho \left[ \left[ f(x) \right]^2 - \left[ g(x) \right]^2 \right] dx \)

**Center of Mass:** \( \overline{X} = \frac{M_y}{M}, \quad \overline{Y} = \frac{M_x}{M} \)
When integrating wrt $y$: $\tilde{x} = \frac{r(y) + l(y)}{2}$, $\tilde{y} = y$ and $dA = [r(y) - l(y)] \, dy$, where $r(y) = \text{right curve}$ and $l(y) = \text{left curve}$ and $\rho(x) = \rho$ because it is a constant for our examples.

**Mass**: $M = \int_{a}^{b} dm = \int_{a}^{b} \rho \left( r(y) - l(y) \right) dy$

**Moment about the $x$-axis**: $M_x = \int_{a}^{b} \rho \, y \left( r(y) - l(y) \right) dy$

**Moment about the $y$-axis**: $M_y = \int_{a}^{b} \rho \frac{(r(y) + l(y))}{2} \left( r(y) - l(y) \right) dy$

$= \frac{1}{2} \int_{a}^{b} \rho \left( [r(y)]^2 - [l(y)]^2 \right) dy$

**Center of Mass**: $\bar{X} = \frac{M_y}{M}$, $\bar{Y} = \frac{M_x}{M}$

3. Centroid

   a. **Def**: If the density function $\delta(x)$ is a constant $\delta$, the center of mass is called a **centroid**.

   b. When the density function is a constant, it cancels out of the numerator and denominator of the formulas for $\bar{X}$ & $\bar{Y}$. So some will use $\delta = 1$, in this case.

   c. When the density function is a constant, the location of the center of mass is independent of the material with which it is made of, but is only dependent on the geometry of the object.

4. Examples
a. Find the centroid ($\rho=1$) of a thin plate covering the region bounded by $y=x-x^2$ and $y=-x$. 
b. Find the center of mass of a planar lamina covering the region between the x-axis, the curve \( y = \frac{2}{x^2}, 1 \leq x \leq 2 \), if the plates density at the point \((x,y)\) is \(\rho = 4\).
c. Find the centroid of a thin plate covering the region bounded by $x=6-y^2$ and $x+2y=3$. 

![Diagram showing the region bounded by $x=6-y^2$ and $x+2y=3$.]
d. Calculate the moments $M_x$ and $M_y$, and the center of mass of a lamina with density $\rho = 3$ and the given shape.