I. Riemann Sums

A. Definition: Given $y = f(x)$: 1. Let $f$ be defined on a closed interval $[a, b]$. 2. Partition $[a, b]$ into $n$ subintervals $[x_{i-1}, x_i]$ of length $\Delta x_i = x_i - x_{i-1}$. Let $P$ denote the partition $a = x_0 < x_1 < \ldots < x_{n-1} < x_n = b$. 3. Let $||P||$ be the length of the longest subinterval. 4. Choose a number $x^*_i$ in each subinterval ($x^*_i$ may be an endpt). 5. Form the Riemann Sum $S_P = \sum f(x^*_i)(\Delta x_i)$.

NOTE: 1. $f$ does not have to be continuous nor nonnegative on $[a, b]$. Therefore, $S_P$ does not necessarily represent an approximation to the area under a graph. 2. $x^*_i$ need not be the same in each interval. 3. $\Delta x_i$ need not be the same length. 4. Riemann Sums are used to approximate a given quantity. 5. To increase the accuracy of the sum, decrease the subinterval length; hence, increase the number of subintervals. 6. As the accuracy of the sum increases, Riemann Sum $\to$ Definite Integral.

B. Examples

1. Finite Sums – finite sums are an example of Riemann Sums in which each subinterval has the same length and the same $x^*_i$ is chosen for each subinterval.

2. Given $y = x^2$ on $[1, 2]$. Partition $[1, 2]$ into the 4 subintervals: $[1, 1.3], [1.3, 1.5], [1.5, 1.6]$ & $[1.6, 2]$. Let $x_1 = 1, x_2 = 1.4, x_3 = 1.6, x_4 = 1.9$. Find the Riemann Sum using this information.

3. Repeat using 4 equal subintervals and $x^*_i$ being the midpoint of each subinterval.
II. The Definite Integral

A. **Def**: If \( f \) is a continuous function defined for \( a \leq x \leq b \), we divide the interval \([a,b]\) into \( n \) subintervals of equal width \( \Delta x = \frac{b-a}{n} \). We let \( x_0 (=a) \), \( x_1 \), \( x_2 \), \ldots , \( x_n (=b) \) be the endpoints of these subintervals and we choose sample points \( x_1^*, x_2^*, \ldots , x_n^* \) in these subintervals, so \( x_i^* \) lies in the \( i \) th subinterval \([x_{i-1}, x_i]\). Then the definite integral of \( f \) from \( a \) to \( b \) is

\[
\int_a^b f(x) \, dx = \lim_{n \to \infty} \sum_{i=1}^{n} \left( f(x_i^*) \Delta x \right)
\]

**NOTE**: 1. The definite integral \( \int_a^b f(x) \, dx \) is a number; it does not depend on \( x \).

\[
\int_a^b f(x) \, dx = \int_a^b f(t) \, dt = \int_a^b f(r) \, dr
\]

2. Because we have assumed that \( f \) is continuous, it can be proved that the limit in the definition always exists and gives the same value no matter how the sample points \( x_i^* \) have been chosen.

3. \( \int_a^b f(x) \, dx \) gives the signed area of a region between the curve \( y=f(x) \) and the \( x \)-axis on \([a,b]\).

B. **Th**: All continuous fns are integrable, i.e., if a fn \( f \) is continuous on \([a,b]\), then its definite integral over \([a,b]\) exists.

C. Examples

1. Express the limit as a definite integral on the given interval

a. \( \lim_{n \to \infty} \sum_{i=1}^{n} \left( 6x_i - 3^{x_i} \right) \Delta x \); \([ -2,3 ] = \)________________________

b. \( \lim_{n \to \infty} \sum_{i=1}^{n} \left[ (x_i)^3 - 7 \right] \Delta x \); \([ 4,7 ] = \)________________________

c. \( \lim_{n \to \infty} \sum_{i=1}^{n} \left( 2 - \left( \frac{4i}{n} \right)^2 \left( \frac{4}{n} \right) \right) = \)________________________
2. Express the definite integrals as a limit similar to the style in example 1c above:
\[
\int_a^b f(x) \, dx = \lim_{n \to \infty} \sum_{i=1}^{n} \left\{ f(x^*_i) \Delta x \right\} ; \quad \text{where} \quad \Delta x = \frac{b-a}{n} \quad \text{and} \quad x^*_i \text{ is the right hand endpoint, i.e., } x^*_i = a + (\Delta x)i .
\]

a. \( \int_4^9 (x^2 - 2x + 3) \, dx \)

b. \( \int_1^4 (x^3 + x) \, dx \)

D. Evaluating Definite Integrals

1. **Approximating the Value of Definite Integrals**

In the previous section we used the area of rectangles \( \sum_{i=1}^{n} \left\{ f(x^*_i) \Delta x \right\} \) to approximate the area under a curve. We now know that the definite integral gives us the “signed” area between a curve and the x-axis. Therefore, we can use this method to approximate the definite integral. If we use midpoints as the \( x^*_i \) value in the definition of a Riemann sum, we call it the Midpoint Rule

a. **Midpoint Rule:** \( \int_a^b f(x) \, dx = \sum_{i=1}^{n} \left\{ f(\bar{x}_i) \Delta x \right\} = \Delta x [f(\bar{x}_1) + f(\bar{x}_2) + \ldots + f(\bar{x}_n)] \)

where \( \Delta x = \frac{b-a}{n} \) and \( \bar{x}_i = \frac{1}{2}(x_{i-1} + x_i) \) = midpoint of \( [x_{i-1}, x_i] \)

b. **Example**

Use the Midpoints Rule with \( n=5 \) to approximate \( \int_1^2 \frac{1}{x} \, dx \)
2. Evaluating the Exact Value of a Definite Integral

a. Using geometry / area of a region to evaluate the exact value of a definite integral

Sometimes the only way to evaluate a definite integral is to use geometry, as in the first example.

1.) \[ \int_{-5}^{5} \sqrt{25-x^2} \, dx \]

2.) \[ \int_{0}^{5} (2-x) \, dx \]

3.) Use the graph of \( g \) below to evaluate \[ \int_{0}^{5} g \, dx \]

b. Using the definition of the definite integral to evaluate the exact value of a definite integral

1.) Background

a.) Sigma Notation for Finite Sums

\[ (1.) \quad \sum_{k=1}^{n} a_k = a_1 + a_2 + a_3 + \ldots + a_n \]
(2.) Examples
(a.) \( \sum_{i=1}^{4} (i^3) = \)

(b.) \( \sum_{j=-2}^{3} (3j + 1) = \)

b.) Sum Formulas for Positive Integers

(1.) \( \sum_{i=1}^{n} i = n(n+1)/2 \)

(2.) \( \sum_{i=1}^{n} i^2 = n(n+1)(2n+1)/6 \)

(3.) \( \sum_{i=1}^{n} i^3 = [n(n+1)/2]^2 \)

(4.) \( \sum_{i=1}^{n} i^4 = \)

c.) Algebra Rules for Finite Sums

(1.) Sum/Difference Rule: \( \sum_{k=1}^{n} (a_k \pm b_k) = \sum_{k=1}^{n} a_k \pm \sum_{k=1}^{n} b_k \)

(2.) Constant Multiple Rule: \( \sum_{k=1}^{n} (ca_k) = c \sum_{k=1}^{n} a_k \), where \( c \) is a constant

d.) Examples

(1.) \( \sum_{k=1}^{10} k^2 - 3k + 2 = \)

(2.) \( \sum_{k=5}^{20} 4k^2 = \)
2. Evaluating a definite integral using the limit definition.

a.) In the definition of the definite integral, \( \int_{a}^{b} f(x) \, dx = \lim_{n \to \infty} \sum_{i=1}^{n} \left[ f(x_i^*) \right] \Delta x \); where \( \Delta x = \frac{b-a}{n} \) we will take \( x_i^* \) to be the right hand endpoint, i.e., \( x_i^* = a + \Delta x i \).

b.) Example

(1.) Evaluate the definite integral using the limit definition.

\[ \int_{-1}^{2} (x^2 + x + 1) \, dx \]
E. Properties of the Definite Integrals

1. Properties

   a. \( \int_{a}^{b} f(x) \, dx = 0 \)

   b. \( \int_{b}^{a} f(x) \, dx = -\int_{a}^{b} f(x) \, dx \)

   c. \( \int_{a}^{b} c \, dx = c \, (b-a) \); \( c \) = constant fn.

   d. \( \int_{a}^{b} k \, f(x) \, dx = k \int_{a}^{b} f(x) \, dx \)

   e. \( \int_{a}^{b} [f(x) \pm g(x)] \, dx = \int_{a}^{b} f(x) \, dx \pm \int_{a}^{b} g(x) \, dx \)

   f. \( \int_{a}^{c} f(x) \, dx + \int_{c}^{b} f(x) \, dx = \int_{a}^{b} f(x) \, dx \)

2. Examples

   Given that \( \int_{1}^{2} f(x) \, dx = 7 \), \( \int_{3}^{1} f(x) \, dx = 5 \), \( \int_{-2}^{2} g(x) \, dx = 3 \) \& \( \int_{2}^{3} g(x) \, dx = -8 \),

   evaluate the following:

   a. \( \int_{-2}^{3} f(x) \, dx = \)

   b. \( \int_{-2}^{3} f(x) \, dx = \)

   c. \( \int_{-2}^{3} -6g(x) \, dx = \)

   d. \( \int_{-2}^{3} [2f(x) + 4g(x)] \, dx = \)

   e. \( \int_{-1}^{2} 5 \, dx \)
F. Comparison Properties of the Integral

1. The Comparison Properties
   a. If \( f(x) \geq 0 \) on \([a,b]\), \( \Rightarrow \int_a^b f(x) \, dx \geq 0 \).
   
   b. If \( f(x) \geq g(x) \) on \([a,b]\), \( \Rightarrow \int_a^b f(x) \, dx \geq \int_a^b g(x) \, dx \).
   
   c. Max-Min Inequality: If \( M \) & \( m \) are the maximum and the minimum values of \( f \) on \([a,b]\), then \( m(b-a) \leq \int_a^b f(x) \, dx \leq M(b-a) \).

2. Example
   Show that the value of \( \int_0^2 \sin(x^2) \, dx \) cannot be 4.