I. Area
   A. Consider the problem of finding the area under the curve on the function $y = -x^2 + 5$ over the domain $[0, 2]$. We can approximate this area by using a familiar shape, the rectangle. If we divide the domain interval into several pieces, then draw rectangles having the width of the pieces, and the height of the curve, we can get a rough idea of the total area.

   For example suppose we divide the interval $[0, 2]$ into 5 equal subintervals of length

   \[ \Delta x = \frac{b - a}{n}, \text{ i.e., each of width } 2/5. \]

   $[0, 0.4], [0.4, 0.8], [0.8, 1.2], [1.2, 1.6], [1.6, 2.0]$

   The table below shows the values obtained when $y(x)$ is evaluated at the corresponding points.

<table>
<thead>
<tr>
<th>$x$</th>
<th>$y = -x^2 + 5$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>5</td>
</tr>
<tr>
<td>0.4</td>
<td>4.84</td>
</tr>
<tr>
<td>0.8</td>
<td>4.36</td>
</tr>
<tr>
<td>1.2</td>
<td>3.56</td>
</tr>
<tr>
<td>1.6</td>
<td>2.44</td>
</tr>
<tr>
<td>2</td>
<td>1</td>
</tr>
</tbody>
</table>

   Plotting these points yields the following graph.

   ![Graph of $y = -x^2 + 5$]

   If we find the minimum value in the subinterval, and use this as our height for that rectangle, we have what is known as an inscribed rectangle. See the graph below.

   ![Inscribed rectangles]

   Now each of the above rectangles has the exact same width, namely 2/5. For this function the height of each rectangle is given by calculating the value of the function at the right hand endpoint of each subinterval. The area under the curve, can then be approximated by adding the areas of all the rectangles together.
Notice that when using the minimum values, i.e. using inscribed rectangles, we arrive at an estimate that is lower than the actual area under the curve. Hence, this method results in what is known as the **lower sum or an underestimate**.

Let's calculate this estimate using the right endpoints

\[
R_5 = \sum_{i=1}^{5} f(x_i) \cdot \Delta x_i
\]

\[
= f(\frac{2}{5}) \cdot \frac{2}{5} + f(\frac{4}{5}) \cdot \frac{2}{5} + f(\frac{6}{5}) \cdot \frac{2}{5} + f(\frac{8}{5}) \cdot \frac{2}{5} + f(2) \cdot \frac{2}{5}
\]

\[
= \frac{2}{5} \left[ f(\frac{2}{5}) + f(\frac{4}{5}) + f(\frac{6}{5}) + f(\frac{8}{5}) + f(2) \right]
\]

\[
= \frac{2}{5} [4.84 + 4.36 + 3.56 + 2.44 + 1]
\]

\[
= \frac{2}{5} [16.2]
\]

\[
= 6.48
\]

You can also calculate an estimate using the maximum value in the subinterval and using it as the height of the rectangles. These rectangles are known as **circumscribed rectangles**. The resulting area approximation will be greater than the area under the curve. Consequently, we call this type of sum an **upper sum or an oversetimate**.

Let's calculate this estimate using the left endpoints

\[
L_5 = \sum_{i=1}^{5} f(x_{i-1}) \cdot \Delta x_i
\]

\[
= f(0) \cdot \frac{2}{5} + f(\frac{2}{5}) \cdot \frac{2}{5} + f(\frac{4}{5}) \cdot \frac{2}{5} + f(\frac{6}{5}) \cdot \frac{2}{5} + f(\frac{8}{5}) \cdot \frac{2}{5}
\]

\[
= \frac{2}{5} \left[ f(0) + f(\frac{2}{5}) + f(\frac{4}{5}) + f(\frac{6}{5}) + f(\frac{8}{5}) \right]
\]

\[
= \frac{2}{5} [5 + 4.84 + 4.36 + 3.56 + 2.44]
\]

\[
= \frac{2}{5} [20.2]
\]

\[
= 8.08
\]

From the two calculations above we can conclude that the area of the curve lies somewhere between the two approximations, i.e. \(6.48 \text{ area of region} < 8.08\)
Another method that can yield a better approximation is known as the **midpoint rule**. In the midpoint rule, you choose the value exactly in the middle of the subinterval to use in calculating the height of the rectangle; resulting in some rectangles being both inscribed & circumscribed.

Let's calculate the above estimate: i.e. the Average or Midpoint Sum.

\[
M_5 = \frac{1}{5} \sum_{i=1}^{5} f \left( x_i^* \right) \cdot \Delta x
\]

\[
= \frac{2}{5} f(1) + \frac{2}{5} f(3) + \frac{2}{5} f(7) + \frac{2}{5} f(9)
\]

\[
= \frac{2}{5} \left[ 4.96 + 4.64 + 4 + 3.04 + 1.76 \right]
\]

\[
= \frac{2}{5} [18.4]
\]

\[
= 7.36
\]

**B. Things to note:**

1. The smaller the subintervals, the better the approximation will be. This is because, the function's values are changing less in the subinterval, i.e. the value of the function is fairly constant in each subinterval. Consequently, we are not approximating by such a rough amount each time. For example, here is the same region divided into 20 rectangle instead of 5. Note that the error is minute compared with the previous work.

a. Each of the above processes (lower sum, upper sum, midpoint sum) are just approximations. They are not exact.

b. When you want to calculate the Volume of a solid, you can use similar techniques, only you'll be using rectangular solids or cylinders to approximate the volume.
2. Defn: The area $A$ of the region $S$ that lies under the graph of the continuous fn $f$ is the limit of the sum of the areas of approximating rectangles:

$$A = \lim_{n \to \infty} R_n = \lim_{n \to \infty} \sum_{i=1}^{n} f(x_i) \cdot \Delta x = \lim_{n \to \infty} \left[ f(x_1) \Delta x + f(x_2) \Delta x + \ldots + f(x_n) \Delta x \right],$$

where $\Delta x = \frac{b-a}{n}$, $x_i = a + i \cdot \Delta x$.

a. Examples

1.) Use the Defn above to find an expression for the exact area under $y=-x^2+5$ on the interval $[0,2]$.

$$2.) \text{ Use the Defn above to find an expression for the exact area under } f = \sqrt{x} \text{ on the interval } [1,4].$$

C. Accuracy

Error Magnitude = |true value - calculated value|

Relative Error = \frac{|true value - calculated sum|}{true value}

Percentage Error = \frac{|true value - calculated sum|}{true value} \times 100\%

For the example in part I, the true value for the area under the curve $y=-x^2+5$ over the domain $[0, 2]$ is $\frac{22}{3} = 7.33$. Therefore the error associated with the approximations

$$\text{Error}_{\text{lower sum}} = \left[ \frac{6.48}{2} \right] \quad 0.11636364$$

$$\text{Error}_{\text{upper sum}} = \left[ \frac{8.08}{2} \right] \quad 0.1018182$$

are:

$$\text{Error}_{\text{midpoint sum}} = \left[ \frac{7.36}{2} \right] \quad 0.0036364$$
II. Distance

A. Constant Velocity

If the velocity of an object remains constant, then the distance can be computed by
\[ \text{distance} = \text{velocity} \times \text{time} \]

B. Variable velocity

1. If the velocity varies, i.e., the object moves with velocity, \( v = f(t) \) where \( a < t < b \) and \( f(t) \geq 0 \), then we will think of the velocity as a constant on each subinterval. If the times are equally spaced, then \( \Delta t = \frac{b - a}{n} \).

   Using the left endpoints, the total distance = \( \sum_{i=1}^{n} f(t_i) \cdot \Delta t \).

   Using the right endpoints, the total distance = \( \sum_{i=1}^{n} f(t_i) \cdot \Delta t \).

   Using the midpoints, the total distance = \( \sum_{i=1}^{n} f(t_i^*) \cdot \Delta t \).

2. This can be thought of as finding the area under the velocity curve where the base of the rectangle is \( \Delta t = \frac{b - a}{n} \) and the height of the rectangle is \( v = f(t) \).

3. Defn: distance = \( A = \lim_{n \to \infty} \sum_{i=1}^{n} f(t_i) \cdot \Delta t = \lim_{n \to \infty} \sum_{i=1}^{n} f(t_i) \cdot \Delta t \)

C. Example

1. A radar gun was used to record the speed of a runner at the times in the table. Estimate the distance the runner covered during those 5 seconds.

<table>
<thead>
<tr>
<th>t (s)</th>
<th>v (m/s)</th>
<th>t (s)</th>
<th>v (m/s)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>3.0</td>
<td>10.5</td>
</tr>
<tr>
<td>0.5</td>
<td>4.67</td>
<td>3.5</td>
<td>10.67</td>
</tr>
<tr>
<td>1.0</td>
<td>7.34</td>
<td>4.0</td>
<td>10.76</td>
</tr>
<tr>
<td>1.5</td>
<td>8.86</td>
<td>4.5</td>
<td>10.81</td>
</tr>
<tr>
<td>2.0</td>
<td>9.73</td>
<td>5.0</td>
<td>10.81</td>
</tr>
<tr>
<td>2.5</td>
<td>10.22</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Using left hand endpoints
Using right hand endpoints

Using midpoints

III. Additional Example
A. Uneven subintervals
  1. Given $y=x^3$ on $[1,3]$. Use the table below to estimate the area between the curve and the x-axis using the left endpoints.

<table>
<thead>
<tr>
<th>$x$</th>
<th>$y=x^3$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>1.4</td>
<td>2.744</td>
</tr>
<tr>
<td>1.6</td>
<td>4.096</td>
</tr>
<tr>
<td>2.1</td>
<td>9.261</td>
</tr>
<tr>
<td>2.2</td>
<td>10.648</td>
</tr>
<tr>
<td>2.5</td>
<td>15.625</td>
</tr>
<tr>
<td>3</td>
<td>27</td>
</tr>
</tbody>
</table>

B. If the function is not strictly increasing nor decreasing.
1. The table below gives the velocity at the specified time. Use this data to give an underestimation of the distance traveled.

<table>
<thead>
<tr>
<th>time (s)</th>
<th>0</th>
<th>2</th>
<th>4</th>
<th>6</th>
<th>8</th>
<th>10</th>
</tr>
</thead>
<tbody>
<tr>
<td>velocity (ft/s)</td>
<td>0</td>
<td>6.1</td>
<td>12.5</td>
<td>8.3</td>
<td>4.9</td>
<td>0</td>
</tr>
</tbody>
</table>