6.1 Area between curves

6.1.1 Area between the curve and the x-axis

**Definition 6.1.** Let \( f(x) \geq 0 \) be continuous on \([a, b]\). The area of the region between the graph of \( f(x) \) and the x-axis is

\[
A = \int_{a}^{b} f(x) \, dx
\]

Let \( f(x) \leq 0 \) be continuous on \([a, b]\). The area of the region between the graph of \( f(x) \) and the x-axis is

\[
A = -\int_{a}^{b} f(x) \, dx
\]

**Example 6.1.1.** Set up the integral(s) needed to find the area of the region bounded by

1) \( y = x^3 \), the x-axis and \( x = 1 \) and \( x = 2 \).

2) \( y = x^2 - x - 2 \) and the x-axis.
3) \( y = 1 - x^2 \) and the \( x \)-axis on \([0, 2]\).

6.1.2 Overview of Area of a Region Between Two Curves

With a few modifications the area under a curve represented by a definite integral can be extended to find the area between two curves. Observe the following graphs of \( f(x) \) and \( g(x) \). Since both graphs lie above the \( x \)-axis, we can geometrically interpret the area of the region between the graphs as the area of the region under \( f(x) \) minus the area of the region under \( g(x) \).

Area of the region between \( f(x) \) and \( g(x) \)

Area of the region under \( g(x) \)

Area of the region under \( f(x) \)
Although we have just shown one case, i.e. when $f$ and $g$ are both positive, this condition is not necessary. All we need in order to evaluate the integrand $[f(x) - g(x)]$ is that $f(x)$ and $g(x)$ both be continuous and $g(x) \leq f(x)$ on the interval $[a, b]$.

6.1.3 Area of a Region Between Two Curves

Finding the Area Between the Curves Using Vertical Rectangles; Integrating with respect to $x$.

If $f(x)$ and $g(x)$ are continuous on $[a, b]$ and $g(x) \leq f(x)$ for all $x$ in $[a, b]$, then the area of the region bounded by the graphs of $f(x)$ and $g(x)$ and the vertical lines $x = a$ and $x = b$ is

$$A = \int_a^b [f(x) - g(x)] \, dx$$

where $f(x)$ is the upper curve and $g(x)$ is the lower curve.

In the next picture, you can see that we simply partition the interval $[a,b]$ and draw our representative rectangles. Let’s drawn a few rectangles to illustrate the concept.

![Diagram of a region between two curves]

**Steps to Find the Area Between Two Curves integrating with respect to $x$**

1. Graph the curves and draw a representative rectangle. This reveals which curve is $f(x)$ (upper curve) and which is $g(x)$ (lower curve). It also helps find the limits of integration if you do not already know them.

2. Find the limits of integration; you may need to find the pts of intersection.

3. Write a formula for $f(x) - g(x)$. Simplify it if you can.

4. Integrate $[f(x) - g(x)]$ from $a$ to $b$. The number you get is the area.

**NOTE:** If you have very complicated functions you can use a graphing calculator, Matlab or another computer algebra system to draw the functions, this will greatly speed up your work. The main reason for doing step 1 is to see how many times the graphs cross (if ever) in the specified domain. It also is to determine which function is dominant in which intervals. If you don’t have a quick means to graph the functions, you can set the two functions equal to each other and try to solve for where they are equal. This will give you the intersection points of the two graphs. Once you know these, you can determine which function is dominant in which subintervals. Then just proceed as before.
Example 6.1.2. Find the indicated areas

1) Find the area of the region bounded by $y = x^2$ and $y = x + 2$.

2) Find the area of the region bounded by $y = 1 - x^2$, $y = (x - 1)^2$ and $y = 1$. 
3) Find the area of the region bounded by \( y = x^3, y = 2 - x^2, x = 0 \) and \( x = 2 \).
Finding the Area Between the Curves Using Horizontal Rectangles; Integrating with respect to $y$.

The procedure of adding rectangles works no matter how we draw the rectangles. We can just of easily draw our rectangles horizontally and integrate with respect to $y$, (your representative rectangles will now be parallel to the $x$ - axis instead of the $y$ - axis).

Consider the following graphs, we will draw horizontal rectangles in one and vertical in the other:

In the first picture, we only need to integrate one time. In the second picture, we would have to integrate twice using different limits of integration and different functions.

Finding the Area Between Two Curves integrating with respect to $y$

If $h(y)$ and $k(y)$ are continuous on $[c, d]$ and $k(y) \leq h(y)$ for all $y$ in $[c, d]$, then the area of the region bounded by the graphs of $h(y)$ and $k(y)$ and the horizontal lines $y = c$ and $y = d$ is

$$
A = \int_c^d [h(y) - k(y)] dy;
$$

where $h(y)$ is the right curve and $k(y)$ is the left curve.

Steps to Finding the Area Between Two Curves integrating with respect to $y$

1. Graph the curves and draw a representative rectangle. This reveals which curve is $h(y)$ (right curve) and which is $k(y)$ (left curve). It also helps find the limits of integration if you do not already know them.

2. Find the limits of integration; you may need to find the points of intersection.

3. Write a formula for $h(y) - k(y)$. Simplify it if you can.

4. Integrate $[h(y) - k(y)]$ from $c$ to $d$. The number you get is the area.
**Example 6.1.3.** Find the indicated areas

1) Set up the integrals to find the area of the region bounded by $x + y^2 = 4$ and $x - y = 2$.

2) Set up the integrals to find the area of the region bounded by $y = 1 - x^2$, $y = (x - 1)^2$ and $y = 1$. 
6.2 Volume

6.2.1 Volume By Slicing

We will be trying to find the volume of a solid shaped using the sum of cross section areas times a width. We will be driving toward developing a Riemann Sum so that we can transform them into integrals.

Definition 6.2. The volume of a solid of known integrable cross-section area \( A(x) \) from \( x = a \) to \( x = b \) is the integral of \( A(x) \) from \( a \) to \( b \),

\[
V = \int_a^b A(x) \, dx.
\]

Steps to Find Volumes by the Method of Slicing

1. Sketch the solid and a typical cross section.
2. Find a formula for \( A(x) \), the area of a typical cross section.
3. Find the limits of integration.
4. Integrate \( A(x) \) to find the volume, \( V = \int_a^b A(x) \, dx \).

Example 6.2.1. Find the volumes of the solids if the solid lies between planes perpendicular to the \( x \)-axis at \( x = 0 \) and \( x = 4 \). The cross sections perpendicular to the \( x \)-axis between these planes run from the parabola \( y = -\sqrt{x} \) to the parabola \( y = \sqrt{x} \). The cross sections are semi-circular disks with diameters in the \( xy \)-plane. Note: the area of a semicircular disk is given by \( A = \frac{\pi r^2}{2} \).

Example 6.2.2. Find the volume of the solid that lies between planes perpendicular to the \( x \)-axis at \( x = -1 \) and \( x = 1 \). In each case, the cross sections perpendicular to the \( x \)-axis, between these planes, run from the semicircle \( y = -\sqrt{1-x^2} \) to the semicircle \( y = \sqrt{1-x^2} \).

1) Assume the cross sections are squares with bases in the \( xy \)-plane.

2) The cross sections are squares with diagonals in the \( xy \)-plane. (The length of a square’s diagonal is \( \sqrt{2} \) times the length of its sides)
Example 6.2.3. The base of the solid is the region between the curve \( y = 2\sqrt{\sin(x)} \) and the interval \([0, \pi]\) on the \( x \)-axis. The cross sections perpendicular to the \( x \)-axis are vertical equilateral triangles with bases running from the \( x \)-axis to the curve. Note: the area of a triangle is given by \( A = \frac{1}{2}bh \) or \( A = \frac{\sqrt{3}}{4}s^2 \)

Example 6.2.4. The solid lies between the planes perpendicular to the \( x \)-axis at \( x = -\frac{\pi}{3} \) and \( x = \frac{\pi}{3} \). The cross sections perpendicular to the \( x \)-axis are circular disks with diameters running from the curve \( y = \tan x \) to the curve \( y = \sec x \).
6.2.2 Volumes of Solids of Revolution  Disks & Washers

Introduction

In this part of the section, we begin with a known planar shape and rotate that shape about a line. The resulting three dimensional shape is known as a solid of revolution. The line about which we rotate the plane shape, is called the axis of rotation. Certain solids of revolution that can be generated like a cylinder, a cone, or a sphere, we can find their volumes using formulas from geometry. However, when the solid or revolution takes on a non-regular shape, like a spool, a bullet, or a blimp, etc... there are often times no easy geometric volume formulas. So we fall back on integral calculus to compute the volume.

A. Disks Method: Rotation about the $x$ - axis:

1. If we revolve a continuous function $y = R(x)$, on $[a, b]$, and the $x$-axis, about the $x$axis as our axis of revolution, then we have a solid formed (with no hole); the cross section perpendicular to the axis of revolution will be a disk of radius $(R(x) - 0)$ which will have area $A(x) = \pi (\text{radius})^2 = \pi [R(x)]^2$.

2. The volume of the solid generated by revolving about the $x$-axis the region between the $x$-axis and the graph of the continuous function $y = R(x)$, $a \leq x \leq b$, is

$$V = \int_{a}^{b} \pi [R(x)]^2 \, dx$$

3. Steps to find the volume of the solid of revolution
   a. Graph the region.
   b. Determine the axis of revolution.
   c. Determine radius $R(x)$.
   d. Determine the limits of integration (Determined by the region).
   e. Integrate.

Example 6.2.5. Find the volume of the solid generated by revolving the region defined by $y = \sqrt{x}$, $x = 1$ about the $x$-axis.

![Graph of $y = \sqrt{x}$, $x = 1$]
B. **Disks Method:** Rotation about the $y$-axis:

(a) Similarly, we can find the volume of the solid when the region is rotated about the $y$-axis.

(b) The volume of the solid generated by revolving about the $y$-axis the region between the $y$-axis and the graph of the continuous function $x = R(y)$, $c \leq y \leq d$, is 

$$V = \int_a^b \pi [R(y)]^2 \, dy$$

**Example 6.2.6.** Find the volume of the solid generated by revolving the region defined by $y = x^3$, $y = 8$ and $x = 0$ about the $y$-axis.

C. **Disks Method:** Rotation about a line other than one of the axes:

If you are revolving the planar object about one of the lines that bound the function, then you proceed in pretty much the same way, since you still have a solid with no hole. In this type of problem, the radius of the disk is still the distance between the curve and the axis of revolution (i.e., subtract radius $R$ from the axis of revolution or vise versa, whichever gives a positive length).

**Example 6.2.7.** Find the volume of the solid generated by revolving the region defined by $y = 2 - x^2$ and $y = 1$ about $y = 1$. 
D. **Washer Method:**

(a) If the region we revolve to generate the solid does not border on or cross the axis of revolution, the solid will have a hole in it. The cross section perpendicular to the axis of revolution are washers instead of disk.

(b) The dimensions of a typical washer are given: Outer Radius: \( R(x) \) Inner Radius: \( r(x) \)

(c) Thus the area of the washer will be given as:

\[
A(x) = \pi (\text{outer radius})^2 - \pi (\text{inner radius})^2 = \pi [R(x)]^2 - \pi [r(x)]^2
\]

\[
A(x) = \pi ([R(x)]^2 - [r(x)]^2)
\]

(d) The volume of such a solid of revolution would be given as:

\[
V(x) = \int_{a}^{b} \pi ([R(x)]^2 - [r(x)]^2) \, dx
\]

(e) This formula can be adjusted to be integrated with respect to \( y \).

(f) The disk formula for finding volume is just the washer formula with \( r(x) = 0 \).

**Example 6.2.8.** Find the volume of the solid generated by revolving the region defined by \( y = x^2 \) and \( y = x \) about the \( x \)-axis.
Example 6.2.9. Find the volume of the solid generated by revolving the region defined by \( y = \sqrt{x}, \ y = 0 \) and \( x = 4 \) about the \( y \)-axis.

Example 6.2.10. Find the volume of the solid generated by revolving the region defined by \( y = x^2 + 2, \ y = \frac{1}{2} x + 1, \ x = 0 \) and \( x = 1 \) about \( y = 3 \).
6.3 Volumes by Cylindrical Shells

6.3.1 Cylindrical shells formula

1. Consider a representative rectangle that has the following characteristics:

\[ w = \text{width of the rectangle} \]
\[ h = \text{height of the rectangle} \]
\[ p = \text{distance between axis of revolution and center of the rectangle}. \]

When this rectangle is revolved about the axis of revolution it forms a cylindrical shell or tube of thickness \( w \). In order to find the volume of this shell, we’ll consider two cylinders, one that has radius \( R(x) = p + \frac{w}{2} \) the other has radius \( r(x) = p - \frac{w}{2} \), since \( p \) is the average radius of the shell.

Recall that the volume of a cylinder is \( V = \pi(\text{radius})^2(\text{height}) = \pi r^2 h \). So the volume of the shell is:

\[
V = \pi R^2 h - \pi r^2 h
= \pi (p + w/2)^2 h - \pi (p - w/2)^2 h
= \pi \left[ (p^2 + pw + w^2/4) - (p^2 - pw + w^2/4) \right] h
= \pi (2pw)h
= 2\pi(\text{average radius})(\text{width})(\text{height})
\]

2. An alternative way to remember formula is to consider a cylindrical shell with outer radius \( r_1 \) and inner radius \( r_2 \). Now imagine cutting and unrolling this cylindrical shell to get a nearly flat rectangular solid. Using the fact that the volume of a rectangular solid is \( V = \text{(length})(\text{width})(\text{height}) \); the volume of this rectangular solid would be \( V = 2\pi r (r_2 - r_1) h = (2\pi r) h \Delta r \). If we now take a solid of revolution and want to approximate its volume by adding the volumes of the shells swept out by \( n \) rectangles and take the limit of this sum as \( \Delta r \rightarrow 0 \) the volume of the solid is:

\[
V = \lim_{\Delta r \to 0} \sum_{k=1}^{n} (2\pi r) h \Delta r = \int_{a}^{b} 2\pi rh \, dr.
\]
6.3.2 Shell Method

A. Rotation about the $y$-axis

1. The volume of the solid generated by revolving the region between the $x$-axis and the graph of the continuous function $y = f(x) > 0$, $a < x < b$ about the $y$-axis, is

$$V = \int 2\pi (\text{shell radius})(\text{shell height}) \, dx$$

$$V = \int_a^b 2\pi xf(x) \, dx$$

2. Steps to find the volume of the solid of revolution
   a. Graph the region.
   b. Determine the axis of revolution.
   c. Determine shell radius.
   d. Determine shell height.
   e. Determine the limits of integration (Determined by the region).
   f. Integrate.

Example 6.3.1. Find the volume of the solid generated by revolving the region defined by $y = 1 + \frac{x^2}{4}$, $y = 0$ from $x = 0$ to $x = 2$ about the $y$-axis.
Example 6.3.2. Find the volume of the solid generated by revolving the region defined by \( y = x^2, \ y = 4 \) from \( x \geq 0 \) about the \( y \)-axis.

B. Rotation about the \( x \)-axis

The volume of the solid generated by revolving the region between the \( y \)-axis and the graph of the continuous function \( x = g(y) \geq 0, \ c \leq x \leq d \) about the \( x \)-axis, is

\[
V = \int_c^d 2\pi (\text{shell radius})(\text{shell height}) \, dy
\]

\[
V = \int_c^d 2\pi yg(y) \, dy
\]

Example 6.3.3. Find the volume of the solid generated by revolving the region in the first quadrant bounded by \( x = y - y^3 \) and \( x = 0 \) about the \( x \)-axis.
C. Rotation about a line other than one of the axes:

**Example 6.3.4.** Find the volume of the solid generated by revolving the region in the first quadrant bounded by \( x = \sqrt[3]{y - 1}, \ y = 9 \) and \( x = 0 \) about:

a. the line \( y = 9 \)

b. the line \( y = -1 \)
**Example 6.3.5.** Find the volume of the solid generated by revolving the region bounded by \( y = x^2 \) and \( y = x + 2 \) about the line \( x = 3 \).

![Diagram of region bounded by curves](image1)

**Example 6.3.6.** Find the volume of the solid generated by revolving the region in the first quadrant bounded by \( y = x^2 \), the \( y \)-axis and the line \( y = 1 \) about the line \( x = 2 \).

![Diagram of region bounded by curves](image2)
6.4 Work

6.4.1 Work Done by a Constant Force

Definition 6.3. If an object is moved a distance \( D \) in the direction of an applied constant force \( F \), then the work \( W \) done by the force is defined as \( W = FD \).

1. Notice that by this definition, if the object you are trying to move, does not move, then no work has been done, even though you have exerted ”work” in your efforts to move it.

2. There are many types of forces - centrifugal, electromotive, and gravitational, etc. We typically measure forces according to their ability to accelerate a unit of mass over time.

Units

<table>
<thead>
<tr>
<th>System of Measurement</th>
<th>Units for Force</th>
<th>Units for Distance</th>
<th>Units for Work</th>
</tr>
</thead>
<tbody>
<tr>
<td>U.S.</td>
<td>pound (lb)</td>
<td>foot (ft)</td>
<td>foot-pounds (ftlb)</td>
</tr>
<tr>
<td></td>
<td></td>
<td>inch (in)</td>
<td>inch-pounds (in-lb)</td>
</tr>
<tr>
<td>S.I.</td>
<td>dyne</td>
<td>centimeters (cm)</td>
<td>erg</td>
</tr>
<tr>
<td></td>
<td>newton (N)</td>
<td>meters (m)</td>
<td>Newton-meter (N-m)</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>Joule (J)</td>
</tr>
</tbody>
</table>

NOTE: 1 joule = (1 newton)(1 meter) = \( 10^7 \) ergs

Conversions

- 1 pound = 4.448 newtons
- 1 slug = 14.59 kilograms
- 1 newton = 0.2248 pounds
- 1 kilogram = 0.06854 slugs
- 1 dyne = 0.02248 pounds
- 1 gram = 0.00006854 slugs

Example 6.4.1. A 150 lb man is climbing a 20 ft pole. Calculate the work done by the man.

6.4.2 Work Done by a Variable Force

Definition 6.4. If an object is moved along a straight line by a continuously varying force \( F(x) \), then the work \( W \) done by the force as the object is moved from \( x = a \) to \( x = b \) is given by:

\[
W = \lim_{n \to \infty} \sum_{i=1}^{n} f(x_i^*) \Delta x = \int_{a}^{b} f(x) \, dx
\]
Example 6.4.2. A mountain climber is about to haul up a 50 m length of hanging rope. How much work will it take if the rope weighs 0.634 N/m?

Example 6.4.3. A 5 lb bucket is lifted from the ground into the air by pulling in 20 ft of rope, weighing 1.6 lbs, over a constant speed. The bucket contains 2 gallons of H$_2$O (16lbs). How much work was spent lifting the H$_2$O, the bucket and the rope?

Example 6.4.4. A 750 ft cable weighing 6 lb/ft is connected to an elevator weighing 1500 lb. Find the work done in lifting the elevator to a height of 500 ft.
6.4.3 Hooke’s Law (Robert Hooke 1635-1703):

**Definition 6.5.** The force $F$ required to compress or to stretch a spring (within its elastic limits) is proportional to the distance $d$ that the spring is compressed or stretched from its original length (natural length). Mathematically speaking: $F = kx$ where the constant of proportionality $k$ (also known as the force constant or spring constant) depends on the specific nature of the spring and $x$ is the number of length units the spring is compressed/stretched from its natural length.

**Example 6.4.5.** A spring of natural length 12” is stretched to 15” under a weight of 10 lb.

a) Find the work done to stretch it from 12” to 13”.

b) Find the work done to stretch it from 15” to 18”.

**Example 6.4.6.** It takes 100 in-lb of work to stretch a spring from its natural length of 10” to 20”.

a) Determine the force constant.

b) Determine the work done to compress the spring from 10” to 8”. 
6.4.4 Pumping Problems

A. The strategy: When finding work done to pump liquids, we need to think of thin slabs of water being removed and then summing those slabs up. Eventually, taking the limit of Riemann Sums to come up with an integral.

B. The Weight of Water: Because of variations in the earth’s gravitational field, the weight of a cubic foot of water at sea level can vary from about 62.26 lbs at the equator to as much as 62.59 lbs near the poles, a variation of about 0.5%. We will use a typical density estimate of $\delta = 62.5 \text{ lbs/ft}^3$ for our calculations. Using the S.I. units of measurement the density of a cubic meter of water is $1000 \text{ kg/m}^3$. Therefore, the estimated weight of the water is $\delta = 9.8 \frac{\text{m}}{\text{s}^2} \times 1000 \frac{\text{kg}}{\text{m}^3} = 9800 \text{ N/m}^3$.

C. Placement of Tank: Whether we place the tank above or below ground does not affect the amount of work done to pump the liquid out of the tank. We will sometimes "place" a tank underground in order to make the algebra easier.

D. Steps in Finding Work Done During Pumping:
   1. Draw a figure with a coordinate system.
   2. Find the volume $\Delta V$ of a thin horizontal slab/slice of liquid.
   3. Find the weight/force $F$ of a thin horizontal slab/slice of liquid. $F = \delta \Delta V$
   4. Find the distance $d$ that the liquid has to travel.
   5. Find the work $\Delta W$ it takes to lift the slab to its destination. $\Delta W = Fd = \delta \Delta V d$
   6. Find the limits of integration. (defined by the region)
   7. Integrate the work expression from the base to the surface of the liquid.

Example 6.4.7. Constant Shaped Tank

1. An aquarium with dimensions 3 x 1 x 2 ft is full of H$_2$O (weighing 62.5 lbs/ft$^3$). Find the work in lifting and pushing the H$_2$O over the top.
2. What if the tank is full of H$_2$O and all the H$_2$O is pumped out 8 feet above the top. (Although the water is moved up a spigot above the top of the tank, for simplicity we disregard the adjustment needed for this change in the receptacle and assume that the entire slab is still being moved up the additional distance.)

**Example 6.4.8.** Variable Shaped Tank

1. Find the work in pumping kerosene, weighing 51.2 lbs/ft$^3$, over the rim of a trough which is 10 ft long and has a semicircular end of diameter 6 ft if the tank is filled to 2/3 of its height.
2. A tank in the form of a right circular cone, vertex down, is filled with H$_2$O to a depth of 1/2 its height. If the height of the tank is 20 ft and its diameter is 8 ft, find the work done pumping all the H$_2$O 5 ft above the top of the tank.

3. To design the interior of a huge stainless steel tank, you revolve the curve $y = x^2$, $0 \leq x \leq 4$, about the y-axis. The container, with dimensions in meters, is to be filled with seawater which weighs 10,000 N/m$^3$. How much work will it take to empty the tank by pumping the seawater to the tanks top?
6.5 Average Value of a Function

6.5.1 Average Value of a Function

Definition 6.6. If \( f(x) \) is integrable on \([a, b]\), its average(mean) value on \([a, b]\) is

\[
\text{f}_{\text{ave}} = \frac{1}{b-a} \int_a^b f(x) \, dx
\]

Example 6.5.1. Find the average value of \( f(x) = \sqrt{x} \) on \([0, 4]\).

6.5.2 Mean Value Theorem for Integral Calculus

Definition 6.7. The Mean Value Theorem for Definite Integrals (MVT): If \( f(x) \) is continuous on \([a, b]\), then at some point \( c \) in \([a, b]\),

\[
f(c) = \frac{1}{b-a} \int_a^b f(x) \, dx
\]

Example 6.5.2. Apply MVT to the example above.

Geometric Interpretation of the MVT

For positive functions \( f(x) \), there is a number \( c \) such that the rectangle with base \([a, b]\) and height \( f(c) \) has the same area as the region under the graph of \( f(x) \) from \( a \) to \( b \).
Example 6.5.3. Find the average value of \( y = \frac{x}{\sqrt{9 + x^2}} \) on \([0, 4]\). Apply the MVT to this problem.

Example 6.5.4. In a certain city the temperature (in \(^\circ\) F) \( t \) hours after 9 a.m. was approximated by the function \( T(t) = 50 + 14 \sin \left( \frac{\pi t}{12} \right) \). Find the average temperature during the period from 9 a.m. to 9 p.m.