4.1 Extrema on an Interval

**Definition 4.1.** Absolute Extreme Values (Global Extreme)
Let \( f(x) \) be a function on an interval \( I \) containing \( x = c \)

1. \( f(c) \) is an absolute maximum value on \( I \) if \( f(c) \geq f(x) \) for all \( x \) in \( I \).
2. \( f(c) \) is an absolute minimum value on \( I \) if \( f(c) \leq f(x) \) for all \( x \) in \( I \).
3. The maximum and minimum values of \( f(x) \) on an interval are called the extreme values of \( f(x) \) on the interval.

**Definition 4.2.** Local Extreme Values (Relative Extreme)
Let \( f(x) \) be a function with domain \( D \) containing \( x = c \)

1. A function \( f(x) \) has a local maximum value at an interior point \( c \) of its domain if \( f(c) \geq f(x) \) for all \( x \) in an open interval containing \( c \).
2. A function \( f(x) \) has a local minimum value at an interior point \( c \) of its domain if \( f(c) \leq f(x) \) for all \( x \) in an open interval containing \( c \).

**Definition 4.3.** Critical Number

1. A critical number of a function \( f(x) \) is a number \( c \) in the domain of \( f(x) \) such that \( f'(c) \) is zero or \( f'(c) \) does not exist.
2. If \( f(x) \) has a local maximum or minimum at \( c \), then \( c \) is a critical number of \( f(x) \).

**Example 4.1.1.** Do the following functions have absolute extrema? Are you guaranteed to have absolute extrema?

![Graph of a function with local and absolute extrema]
Theorem 4.1.1. The Extreme Value Theorem: If \( f(x) \) is continuous on a closed interval \([a, b]\), then \( f(x) \) attains an absolute maximum value \( f(c) = M \) and an absolute minimum value \( f(d) = m \) at some numbers \( c \) and \( d \) in \([a, b]\).

Note: If the hypothesis of this theorem are met, then you are GUARANTEED the existence of a Max or Min value. If the hypothesis are not met, you MAY have a max or min value.

Example 4.1.2. Find the local and absolute extrema for each of the following:

1. \( f(x) = x^2 - 2x \) on \([-1, 3.5]\)

2. \( f(x) = x^2 - 2x \) on \((-1, 3.5]\)

3. \( f(x) = x^2 - 2x \) on \((-1, 3.5)\)
Example 4.1.3. Is there a local minimum or minimum at $x = 3$?

![Graph showing function values]

Theorem 4.1.2. Fermat’s Theorem

If $f(x)$ has a local maximum or minimum value at an interior point $c$ of its domain, and if $f'(x)$ is defined at $c$, then $f(c) = 0$.

Procedure to find absolute extrema

To find absolute extrema for a function $f(x)$ that is continuous on a closed interval $[a, b]$

1. Find all critical numbers for $f(x)$ in $(a, b)$. (i.e. when is $f'(x) = 0$ zero or undefined)
2. Evaluate $f(x)$ at all critical numbers in $(a, b)$.
3. Evaluate $f(x)$ at the endpoints $x = a$ and $x = b$ of the interval $[a, b]$.
4. Chose the largest value found in steps 2 and 3 as the maximum value of $f(x)$ on $[a, b]$ and the smallest as the minimum value of $f(x)$ on $[a, b]$.

Example 4.1.4. Determine the absolute extrema for $f(x) = x^3 - 6x^2 + 1$ on $[-2, 3]$
Example 4.1.5. Determine the absolute extrema for $f(x) = x^{4/3} + 4x^{1/3}$ on $[-2, 2]$.

Example 4.1.6. Determine the absolute extrema for $h(x) = \frac{\ln x}{x}$ on $[1, 3]$. 
4.2 The Mean Value Theorem

**Theorem 4.2.1. Rolle’s Theorem:** Let \( f(x) \) be a function that satisfies the following three hypotheses.

1. \( f(x) \) is continuous on the closed interval \([a, b]\)
2. \( f(x) \) is differentiable on the open interval \((a, b)\)
3. \( f(a) = f(b) \)

Then there is a number \( c \) in \((a, b)\) such that \( f'(c) = 0 \)

**Example 4.2.1.** Verify that \( f(x) = x\sqrt{x+6} \) satisfies the three hypotheses of Rolle’s Theorem on the interval \([-6, 0]\). Then find all numbers \( c \) that satisfy the conclusion of Rolle’s Theorem.

**Example 4.2.2.** The function \( f(x) = \begin{cases} x & \text{if } 0 \leq x < 1 \\ 0 & \text{if } x = 0 \end{cases} \) is zero at \( x = 0 \) and \( x = 1 \) and differentiable on \((0, 1)\), but its derivative on \((0, 1)\) is never zero. How can this be? Explain why this doesn’t violate Rolle’s theorem.
Theorem 4.2.2. The Mean Value Theorem: Let $f(x)$ be a function that satisfies the following hypotheses:

1. $f(x)$ is continuous on the closed interval $[a, b]$
2. $f(x)$ is differentiable on the open interval $(a, b)$

Then there is a number $c$ in $(a, b)$ such that: $f'(c) = \frac{f(b) - f(a)}{b - a}$
or, equivalently: $(b - a)f'(c) = f(b) - f(a)$

Graphical interpretation:

1. The Mean Value Theorem says that at some interior point the instantaneous rate of change must equal the average rate of change.
2. There is an interior point where the slope of the tangent line must be equal to the slope of the secant line. i.e. The tangent line is parallel to the secant line.

Example 4.2.3. The graph of $f(x) = x + \frac{4}{x}$ is below. Graph the secant line through the points $(1, 5)$ and $(8, 8.5)$ on the graph.

Find the number $c$ that satisfies the conclusion of the Mean Value Theorem for this function $f(x)$ and the interval $[1,8]$. Then graph the tangent line at the pt $(c, f(c))$ and notice that it is parallel to the secant line.
Example 4.2.4. Verify that \( f(x) = e^{-2x} \) satisfies the hypotheses of the Mean Value Theorem on the interval \([0, 3]\). Then find all numbers \( c \) that satisfy the conclusion of the Mean Value Theorem.

Some other Theorems and Corollaries

**Theorem:** If \( f'(x) = 0 \) for all \( x \) in an interval \((a, b)\), then \( f(x) \) is a constant. **Theorem:** If \( f'(x) = g'(x) \) for all \( x \) in an interval \((a, b)\), then \( f(x) - g(x) \) is constant on \((a, b)\). i.e. \( f(x) = g(x) + c \) where \( c \) is a constant.

Example 4.2.5. Show that the equation \( 3x - 2 + \cos \left( \frac{\pi x}{2} \right) = 0 \) has exactly one real root.

Example 4.2.6. For what values of \( a, m, \) and \( b \) does the function \( f(x) = \begin{cases} x & \text{if } x = 0 \\ -x^2 + 3x + a & \text{if } 0 < x < 1 \\ mx + b & \text{if } 1 \leq x \leq 2 \end{cases} \) satisfy the hypotheses of the Mean Value Theorem on the interval \([0, 2]\).
4.3 How Derivatives Affect the Shape of a Graph

4.3.1 Determining the Intervals Where a Function is Increasing or Decreasing

**Definition 4.4.** Let \( f(x) \) be a function defined on an interval \( I \) and let \( x_1 \) and \( x_2 \) be any two points in \( I \).

1. \( f(x) \) is **increasing** on \( I \) if \( x_1 < x_2 \), then \( f(x_1) < f(x_2) \)

2. \( f(x) \) is **decreasing** on \( I \) if \( x_1 > x_2 \), then \( f(x_1) > f(x_2) \)

**The first derivative test for increasing/decreasing.**

Suppose that \( f(x) \) is continuous on \([a, b]\) and differentiable on the open interval \((a, b)\).

- If \( f'(x) > 0 \) for all \( x \) in \((a, b)\) then \( f(x) \) increases on \([a, b]\).
- If \( f'(x) < 0 \) for all \( x \) in \((a, b)\) then \( f(x) \) decreases on \([a, b]\).
- If \( f'(x) = 0 \) for all \( x \) in \((a, b)\) then \( f(x) \) is constant on \([a, b]\).

4.3.2 Local Extrema (Relative Extrema)

Definitions:

1. A function \( f(x) \) has a **local maximum value** at a point \( c \) if it is the highest point near itself.

2. A function \( f(x) \) has a **local minimum value** at a point \( c \) if it is the lowest point near itself.

3. A **critical number** of a function \( f(x) \) is a number \( c \) in the domain of \( f(x) \) such that \( f'(x) \) is zero or \( f'(x) \) does not exist.

**The First Derivative Test for Local Extrema.**

Let \( f(x) \) be a continuous function on \([a, b]\) and \( c \) be a critical number in \([a, b]\).

1. If \( f'(x) \geq 0 \) on \((a, c)\) and \( f'(x) \leq 0 \) on \((c, b)\), then \( f(x) \) has a local maximum of \( y = f(c) \) at \( x = c \).

2. If \( f'(x) \leq 0 \) on \((a, c)\) and \( f'(x) \geq 0 \) on \((c, b)\), then \( f(x) \) has a local minimum of \( y = f(c) \) at \( x = c \).

3. If \( f'(x) \) does not change signs at \( x = c \), then \( f(x) \) has no local extrema at \( x = c \).
### 4.3.3 Determining the Intervals of Concavity

**Definition 4.5.** The graph of a differentiable function \( y = f(x) \) is **concave up** on an interval where \( f'(x) \) is increasing and **concave down** on an interval where \( f'(x) \) is decreasing.

**Question:** How do find where \( f'(x) \) is increasing or decreasing?

**Answer:** The same way we did if for \( f(x) \). Take the derivative of \( f'(x) \) and see where it is positive (increasing) and negative (decreasing).

If we take the derivative of the derivative we have found the second derivative. So now we have the second derivative test for concavity:

**The Second Derivative Test for Concavity**

Let \( f(x) \) be a twice differentiable function on an interval \( I \).

1. If \( f''(x) > 0 \) on \( I \), the graph of \( f(x) \) over \( I \) is concave up.
2. If \( f''(x) < 0 \) on \( I \), the graph of \( f(x) \) over \( I \) is concave down.

**Example 4.3.1.** Draw an example of the following:

<table>
<thead>
<tr>
<th>Concave up &amp;</th>
<th>Concave up &amp;</th>
<th>Concave down &amp;</th>
<th>Concave down &amp;</th>
</tr>
</thead>
<tbody>
<tr>
<td>Decreasing</td>
<td>Increasing</td>
<td>Decreasing</td>
<td>Increasing</td>
</tr>
</tbody>
</table>

**Definition 4.6.** An **inflection point** is a point of the graph where the function changes from concave up to concave down or down to up. We can find inflection points with the second derivative:

Find where \( f''(x) = 0 \) and/or where \( f''(x) \) does not exist.

**The Second Derivative Test for Local Extrema**

Let \( f(x) \) be a continuous function on \([a, b]\) and \( c \) be a critical point in \([a, b]\).

1. Compute \( f'(x) \) and \( f''(x) \).
2. Find all the critical numbers of \( f \) at which \( f'(x) = 0 \).
3. Compute \( f''(c) \) for each such critical number \( c \).

   (a) If \( f''(c) < 0 \), then \( f(x) \) has a local maximum of \( y = f(c) \) at \( x = c \).
   (b) If \( f''(c) > 0 \), then \( f(x) \) has a local minimum of \( y = f(c) \) at \( x = c \).
   (c) If \( f''(c) = 0 \), then the test is inconclusive and you must use the First Derivative Test for Local Extrema.
Example 4.3.2. Below is the graph of a function $f(x)$.

![Graph of f(x)](image1)

a) $f(x)$ is increasing on ________________  
b) $f(x)$ is decreasing on ________________

c) $f(x)$ has a local maximum at $x =$ _____  
d) $f(x)$ has a local minimum at $x =$ _____

e) $f(x)$ is concave up on ________________  
f) $f(x)$ is concave down on ________________

g) $f(x)$ has inflection point(s) at $x =$ __________

Example 4.3.3. Below is the graph of a function $f'(x)$.

![Graph of f'(x)](image2)

a) $f(x)$ is increasing on ________________  
b) $f(x)$ is decreasing on ________________

c) $f(x)$ has a local maximum at $x =$ _____  
d) $f(x)$ has a local minimum at $x =$ _____

e) $f(x)$ is concave up on ________________  
f) $f(x)$ is concave down on ________________

g) $f(x)$ has inflection point(s) at $x =$ __________
Example 4.3.4. Below is the graph of a function $g'(x)$.

![Graph of $g'(x)$]

a) $g(x)$ is increasing on ________________

b) $g(x)$ is decreasing on ________________

c) $g(x)$ has a local maximum at $x = _____$

d) $g(x)$ has a local minimum at $x = _____$

e) $g(x)$ is concave up on ________________

f) $g(x)$ is concave down on ________________

g) $g(x)$ has inflection point(s) at $x = _________$

Example 4.3.5. Below is the graph of a function $f'(x)$.

![Graph of $f'(x)$]

a) $f(x)$ is increasing on ________________

b) $f(x)$ is decreasing on ________________

c) $f(x)$ has a local maximum at $x = _____$

d) $f(x)$ has a local minimum at $x = _____$

e) $f(x)$ is concave up on ________________

f) $f(x)$ is concave down on ________________

g) $f(x)$ has inflection point(s) at $x = _________$
Example 4.3.6. Given the following information, graph the function \( f(x) \)

(i) \( f'(x) > 0 \) for \(-1 < x < 1\), and \( x > 3\).

(ii) \( f'(x) < 0 \) for \( x < -1\), and \( 1 < x < 3\).

(iii) \( f''(x) > 0 \) for \( x < 0\), and \( x > 2\).

(iv) \( f''(x) < 0 \) for \( 0 < x < 2\).
4.5 Summary of Curve Sketching

Review

A **Horizontal Asymptote** describes the behavior of a function as \( x \) gets very large.

**Limits at Infinity/Horizontal Asymptotes**

Let \( f(x) \) be the rational function given by

\[
f(x) = \frac{N(x)}{D(x)} = \frac{a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0}{b_m x^m + b_{m-1} x^{m-1} + \cdots + b_1 x + b_0}
\]

where \( N(x) \) and \( D(x) \) have no common factors. The \( \lim_{x \to \infty} f(x) \) can be determined by comparing the degrees of \( N(x) \) and \( D(x) \).

1. If \( n < m \), then \( \lim_{x \to \infty} f(x) = 0 \) and the graph of \( f(x) \) has the line \( y = 0 \) (the \( x \)-axis) as a horizontal asymptote.

2. If \( n = m \) then \( \lim_{x \to \infty} f(x) = \frac{a_n}{b_m} \) and the graph of \( f(x) \) has the line \( y = \frac{a_n}{b_m} \) as a horizontal asymptote.

3. If \( n > m \) then \( \lim_{x \to \infty} f(x) = \pm \infty \) and the graph of \( f(x) \) has no horizontal asymptote.

**Vertical Asymptotes**

The line \( x = a \) is a **vertical asymptote** of \( f \) if either

\[
\lim_{x \to a^+} f(x) = \pm \infty
\]

OR

\[
\lim_{x \to a^-} f(x) = \pm \infty
\]

You find a vertical asymptote of a function \( f(x) = \frac{N(x)}{D(x)} \) by finding a value \( x = a \) such that denominator equals zero \( D(a) = 0 \) AND the numerator is not zero \( N(a) \neq 0 \).
The first derivative test for increasing/decreasing.
Suppose that \( f(x) \) is continuous on \([a, b]\) and differentiable on the open interval \((a, b)\).
If \( f'(x) > 0 \) for all \( x \) in \((a, b)\) then \( f(x) \) increases on \([a, b]\).
If \( f'(x) < 0 \) for all \( x \) in \((a, b)\) then \( f(x) \) decreases on \([a, b]\).
If \( f'(x) = 0 \) for all \( x \) in \((a, b)\) then \( f(x) \) is constant on \([a, b]\).

The First Derivative Test for Local Extrema.
Let \( f(x) \) be a continuous function on \([a, b]\) and \( c \) be a critical number in \([a, b]\).

1. If \( f'(x) \geq 0 \) on \((a, c)\) and \( f'(x) \leq 0 \) on \((c, b)\), then \( f(x) \) has a local maximum of \( y = f(c) \) at \( x = c \).
2. If \( f'(x) \leq 0 \) on \((a, c)\) and \( f'(x) \geq 0 \) on \((c, b)\), then \( f(x) \) has a local minimum of \( y = f(c) \) at \( x = c \).
3. If \( f'(x) \) does not change signs at \( x = c \), then \( f(x) \) has no local extrema at \( x = c \).

The Second Derivative Test for Concavity
Let \( f(x) \) be a twice differentiable function on an interval \( I \).

1. If \( f''(x) > 0 \) on \( I \), the graph of \( f(x) \) over \( I \) is concave up.
2. If \( f''(x) < 0 \) on \( I \), the graph of \( f(x) \) over \( I \) is concave down.

Graphing using \( y' \) and \( y'' \):

1. Determine the points of discontinuity.
2. Determine the asymptotes (vertical, horizontal)
3. Determine the \( x \)- and \( y \)-intercepts.
4. Determine the critical point(s). (Set \( f'(x) = 0 \) and undefined).
5. Determine the intervals where the function \( f \) is increasing/decreasing.
6. Determine the local extrema.
7. Determine the possible point(s) of inflection. Set \( f''(x) = 0 \) and undefined).
8. Determine the intervals where the function \( f \) is concave up/down.
9. Determine the inflection point(s).
10. Determine extra point(s) if necessary.
11. Sketch the graph using the information obtained above.
Example 4.5.1. Graph $f(x) = x^3 + 2x^2 + 1$ using the steps above.

Example 4.5.2. Graph $y = x^{2/3}$ using the steps above.
Example 4.5.3. \( g(x) = \frac{6x^2 - 27x + 12}{2x^2 - 7x - 4} = \frac{3(2x - 1)(x - 4)}{(2x + 1)(x - 4)} \); \( g'(x) = \frac{12}{(2x + 1)^2} \); \( g''(x) = \frac{-48}{(2x + 1)^3} \).
Example 4.5.4. Below is the graph of $y = f'(x)$.

![Graph of f'(x)](image)

a) Find all critical points of $f(x)$

b) On what intervals does the graph of $f(x)$ increase? decrease? Write in words how you would find this using the graph of $f'(x)$.

c) Can you determine where $f''(x)$ is positive? Negative? Write out in words how you would find this using the graph of $f'(x)$.

d) Where is $f(x)$ concave up? Concave down?

e) Sketch a possible graph of $f(x)$. 
4.7 Optimization Problems

Steps for Solving Word Problems:

1. Read.
2. Draw a picture when applicable.
3. Determine if you are maximizing or minimizing the problem.
4. Summarize the information in the problem statement.
5. Determine the formula/function that applies.
6. Write the function in terms of one variable.
7. Determine the domain of the function.
8. Determine the critical point(s)
9. Test to determine the extrema (section 4.1).
10. Did you answer the question asked?

Example 4.7.1. What is the largest possible area for a right triangle whose hypotenuse is 5 cm long?
Example 4.7.2. Farmer Bob has 200 ft. of fencing to enclose a rectangular field. What is the largest possible area that he can enclose if he makes 2 side by side corrals by adding a piece of fencing parallel to the shorter side?

Example 4.7.3. Four feet of wire is to be used to form a square and a circle. How much of the wire should be used for the square and how much should be used for the circle to enclose the maximum area?
Example 4.7.4. An open box with capacity of 36,000 cubic inches is needed. If the box must be twice as long as it is wide, what dimensions would require the least amount of material?

Example 4.7.5. A metal cylindrical container with a closed top is to hold $1\text{ft}^3$. Find the dimensions which require the least amount of material.
4.4 L'Hopital’s Rule

Review: Evaluate the following

1. \[ \lim_{x \to 2} \left( \frac{x - 2}{x - 3} \right) \]
2. \[ \lim_{x \to 3} \left( \frac{x - 3}{x^2 - 9} \right) \]
3. \[ \lim_{x \to 0} \left( \frac{\sin x}{x} \right) \]
4. \[ \lim_{x \to 0^+} \left( \frac{e^x}{x} \right) \]
5. \[ \lim_{x \to 0^+} \left( \frac{\ln x}{\cot x} \right) \]

Indeterminate and Determinate Forms

1. Indeterminate forms occur when we have equations whose limits involve a form that looks like:
   \[ \frac{0}{0}, \quad \infty, \quad -\infty - \infty, \quad 0^0, \quad \infty^0, \quad 1^\infty. \]
   Here indeterminate means we don’t know if the limit exists or not.

2. Determinate forms are of the following types:
   \[ \infty + \infty \to \infty, \quad -\infty - \infty \to -\infty, \quad 0^\infty \to 0, \quad 0^{-\infty} \to \infty \]

L’Hospital’s Rule

Theorem 4.4.1. L’Hospital’s Rule (Strong Form): Suppose that \( f(x) \) and \( g(x) \) are differentiable on an open interval \( I \) containing \( a \). Suppose also that \( g'(x) \neq 0 \) on \( I \) if \( x \neq a \). Suppose that
   \[ \lim_{x \to a} f(x) = 0 \quad \text{and} \quad \lim_{x \to a} g(x) = 0 \]
   or that
   \[ \lim_{x \to a} f(x) = \pm \infty \quad \text{and} \quad \lim_{x \to a} g(x) = \pm \infty \]
   (In other words, we have an indeterminate form of the type \( \frac{0}{0} \) OR \( \frac{\infty}{\infty} \)). Then
   \[ \lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}, \]
   if the limit on the right exists (or is \( \pm \infty \)).

NOTE: In the strong form it is important that you realize something. We do not know that the first limit is actually equal to the second limit, UNTIL we have shown that the second limit exists or is infinite. In other words if the second limit can not be established then we can not equate the two limits.
Example 4.4.1. Indeterminate quotients: \( \frac{0}{0} \) OR \( \frac{\pm \infty}{\infty} \). Evaluate the following limits.

1. \( \lim_{x \to 3} \left( \frac{x - 3}{x^2 - 9} \right) \)

2. \( \lim_{x \to 0} \left( \frac{\sin x}{x} \right) \)

3. \( \lim_{x \to 0^+} \left( \frac{\ln x}{\cot x} \right) \)

4. \( \lim_{y \to \infty} \left( \frac{y^2}{\ln y} \right) \)

5. \( \lim_{x \to \infty} \left( \frac{e^{-x}}{1 + e^{-x}} \right) \)

6. \( \lim_{t \to 0} \left( \frac{1 - e^{3t} + 3t}{t^2} \right) \)
7. \[ \lim_{y \to 4} \left( \frac{y - 4}{\sqrt[3]{y^4 + 4^2}} \right) \]

Example 4.4.2. Indeterminate products and differences: \( 0 \cdot \infty, \infty - \infty \). You need to use algebra to manipulate the equations so that you get either: \( \frac{0}{0} \) OR \( \pm \frac{\infty}{\infty} \)

1. \[ \lim_{x \to \infty} x \left( e^{1/x} - 1 \right) \]

2. \[ \lim_{\theta \to 0^+} \theta \cot \theta \]

3. \[ \lim_{t \to 0^+} \left( \frac{1}{\sin t} - \frac{1}{t^2} \right) \]
Example 4.4.3. Indeterminate Powers: $0^0$, $\infty^0$, $1^\infty$. You can sometimes write the equations in terms of $e$:

$$\lim_{x \to a} [f(x)]^{g(x)} = \lim_{x \to a} e^{g(x) \ln f(x)} = e^{\lim_{x \to a} g(x) \ln f(x)}$$

1. \[ \lim_{x \to 0^+} x^{3x} \]

2. \[ \lim_{x \to 0^+} (1 + x)^{1/x} \]

3. \[ \lim_{x \to \frac{\pi}{2}^+} (\tan x)^{(x-\pi/2)} \]
4.9 Antiderivatives

4.9.1 Review from chapters 2 & 3:

1. Derivative of \( f(x) \): \[ f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \]

2. Notation: \( \frac{dy}{dx}, \frac{df}{dx}, \frac{d}{dx}[f(x)], \ y', \ f'(x), \ D_x f, \ \dot{y} \)

3. Applications:

   (a) \( f'(c) \) = slope of the tangent line to the graph of \( y = f(x) \) at the point \( (c, f(c)) \)

   (b) \( f'(c) \) = the rate of change of \( y \) with respect to \( x \) when \( x = c \). i.e. how fast a particle moves if \( x = \) time and \( y = \) distance.

4.9.2 Integration

A. Find the derivative of

1. \( F(x) = x^2 \) 
2. \( F(x) = x^2 + 7 \) 
3. \( F(x) = x^2 - 0.75 \)

\[ f'(x) = \quad \quad f'(x) = \quad \quad f'(x) = \]

So, if \( f(x) = 2x \), then \( F(x) = x^2 + C \) is the antiderivative of \( f(x) \)

B.

**Definition 4.7.** A function \( F(x) \) is an antiderivative of a function \( f(x) \), if \( F'(x) = f(x) \) for all \( x \) in the domain of \( f \).

C.

**Theorem 4.9.1.** If \( F \) is an antiderivative of \( f \) on an interval \( I \), then the most general antiderivative of \( f \) on \( I \) is \( F(x) + C \) where \( C \) is an arbitrary constant.

D. The set of all antiderivatives of \( f \) is the indefinite integral of \( f \) with respect to \( x \), denoted by \( f(x) \, dx \).

ie. \[ \int f(x) \, dx = F(x) + C \ \text{IFF} \ F'(x) = f(x). \]

**Note:** Any two antiderivates of a function differ only by a constant.

**Example 4.9.1.** Use your knowledge of derivatives to determine the following,

\[ \int 3x^2 \, dx = \quad \quad \int \cos(x) \, dx = \quad \quad \int 2y \, dy = \]
E. Formulas

1. \( \int k \, dx = kx + C \)
2. \( \int x^n \, dx = \frac{x^{n+1}}{n+1}; \quad n \in \mathbb{Q}, n \neq 1 \)
3. \( \int kf(x) \, dx = k \int f(x) \, dx \)
4. \( \int f(x) \pm g(x) \, dx = \int f(x) \, dx \pm \int g(x) \, dx \)
5. \( \int \frac{1}{x} \, dx = \ln|x| + C \)
6. \( \int e^x \, dx = e^x + C \)

Example 4.9.2. Integrate the following

1. \( \int 12 \, dx = \)
2. \( \int x^3 \, dx = \)
3. \( \int \, dx = \)
4. \( \int \left( e^x + \frac{1}{x} \right) \, dx = \)
5. \( \int \left( 5\sqrt{t^3} + 7 \right) \, dt = \)
6. \( \int (x^3 + 1)^2 \, dx = \)
7. \( \int 2y \, dy = \)

Example 4.9.3. Find the most general antiderivative of the function \( f(x) = \frac{4}{x^6} \).
F. Initial Value Problems

**Definition 4.8.** An equation like \( \frac{dy}{dx} = f(x) \) that has a derivative in it is called a differential equation. The problem of finding a function \( y \) of \( x \) when we know its derivative and an initial value \( y_0 \) at a particular point \( x_0 \) is called an initial value problem.

**Example 4.9.4.** Find the initial function given the initial condition(s).

a. \( \frac{dy}{dx} = e^x + 2x^4; \quad y(0) = 7 \)

b. \( f''(x) = 6x^2 - 10; \quad f(0) = 9; \quad f(2) = -35 \)
c. A ball is thrown upward from the top of a building 160 ft tall. If the ball’s upward velocity at time \( t = 1 \) sec is 16 ft/sec, determine (a) how long it takes the ball to reach the ground below, and (b) the ball’s impact velocity. Ignore air resistance.

d. The graph of \( f' \) is given below, sketch the graph of \( f \) if \( f \) is continuous and \( f(-3) = -6 \).
G. More Formulas

7. \( \int \sin x \, dx = -\cos x + C \)  
8. \( \int \cos x \, dx = \sin x + C \)

9. \( \int \sec^2 x \, dx = \tan x + C \)  
10. \( \int \csc^2 x \, dx = -\cot x + C \)

11. \( \int \sec x \tan x \, dx = \sec x + C \)  
12. \( \int \csc x \cot x \, dx = -\csc x + C \)

13. \( \int \frac{dx}{\sqrt{1 - x^2}} = \sin^{-1} x + C \)  
14. \( \int \frac{dx}{1 + x^2} = \tan^{-1} x + C \)

Example 4.9.5. Integrate the following:

1. \( \int (5 \cos x - 2x^7 + 3e^x) \, dx = \)

2. \( \int (\sin \theta + \sec \theta \tan \theta) \, d\theta = \)

3. \( \int \frac{5dx}{\sqrt{1 - x^2}} = \)

4. Find the antiderivative \( F \) of \( f \) that satisfies the given initial condition.

\[ f(x) = 2x - \frac{3}{1 + x^2}; \quad F(\sqrt{3}) = 1 \]
Example 4.9.6. Given that $v(t) = 4 \cos t$ and that $s(\pi) = 1$, find the object’s position at time $t = \frac{\pi}{4}$ sec.

Example 4.9.7. Given that the graph of $f$ passes through the pt $\left(\frac{\pi}{4}, 3\right)$ and the slope of its tangent at the point $(x, f(x))$ is $\sec^2 x$, find $f$.

Example 4.9.8. A car braked with constant deceleration of $40 \, \text{ft/s}^2$, producing skid marks measuring 160 ft before coming to a stop. How fast was the car traveling when the brakes were first applied?