6.1 Introduction

exponent

base $\rightarrow x^2$, x^3, $x^{1/2} = \sqrt{x}$ $\quad \sqrt{y^2} = \pm\sqrt{4}$

6.1.1 Exponential Functions

$x^{1/7} = \sqrt[7]{x}$ $\quad x = \pm 2$ ✓

The Exponential Function

The exponential function with base a is denoted by

$$f(x) = a^x \qquad 2^x$$

where $\underline{a \neq 0}$, $\underline{a \neq 1}$, and x is any real number.

positive

Graphs of exponential functions

Consider $f(x) = 2^x$ and $g(x) = 4^x$.

Let's look at a table of x and y values for these functions.

negative exponent means the number is on the wrong side of the division line.

$$\frac{1}{2^2} = \frac{2^{-2}}{1}$$

$$\frac{1}{2^{-2}} = \frac{2^2}{1}$$

x	$f(x)$ 2^x	$g(x)$ 4^x
-2	1/4	1/16
-1	1/2	1/4
0	1	1
1	2	4
2	4	16
3	8	64

3, 12, 48

$2^{-2} = \frac{1}{4}$

$2^{-1} = \frac{1}{2}$

$2^0 = 1$

2^1 $\cdot 16$

2^2

2^3

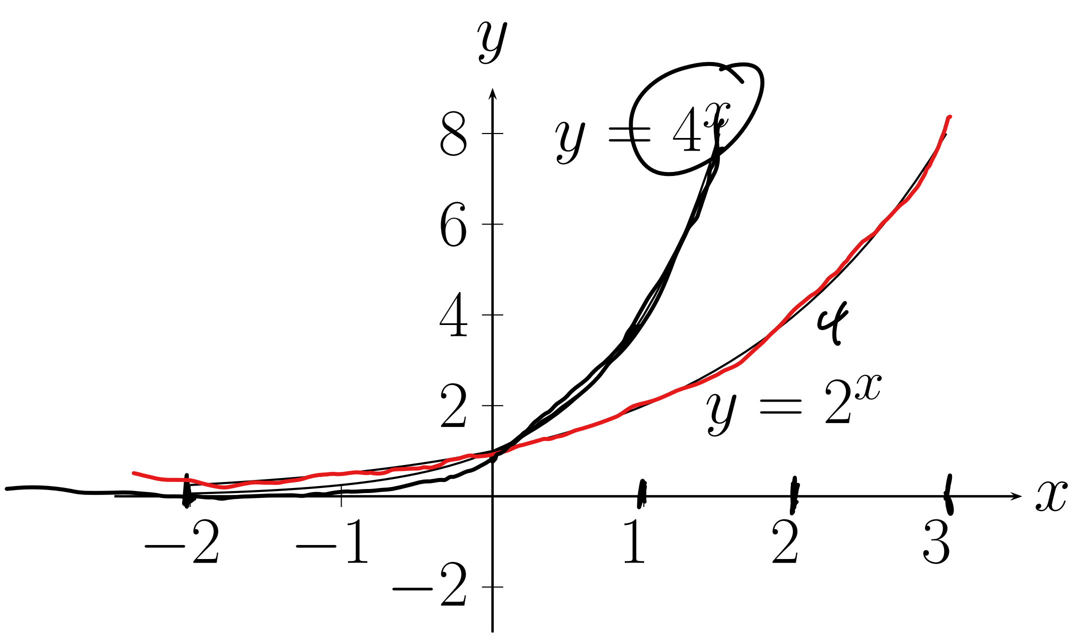
$4^{-2} = \frac{1}{4^2} = \frac{1}{16}$

$4^{-1} = \frac{1}{4}$

$4^0 = 1$ $\qquad 150^0 = 1$

$4^3 = 64$

We can graph this data on the xy-axes.



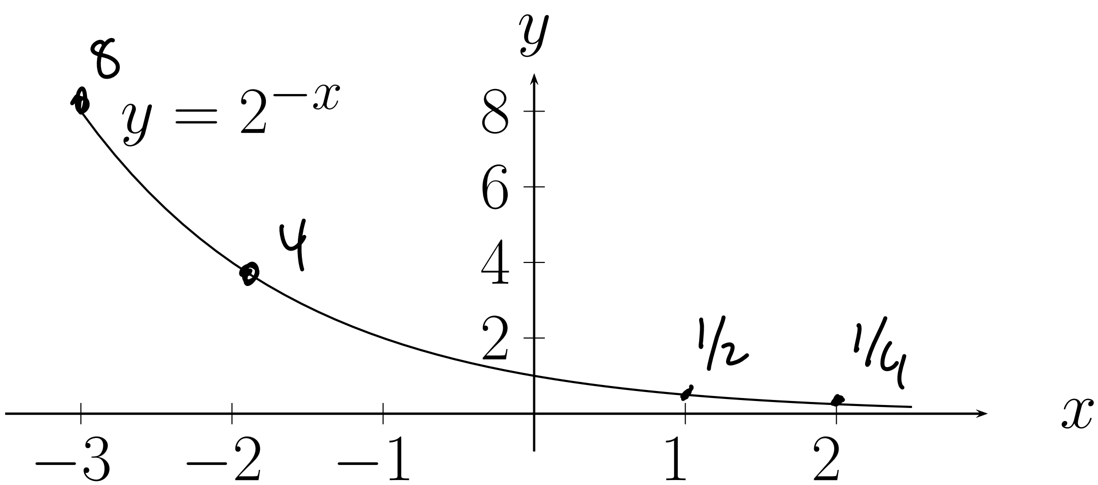
Can't get negatives

Can't get zero

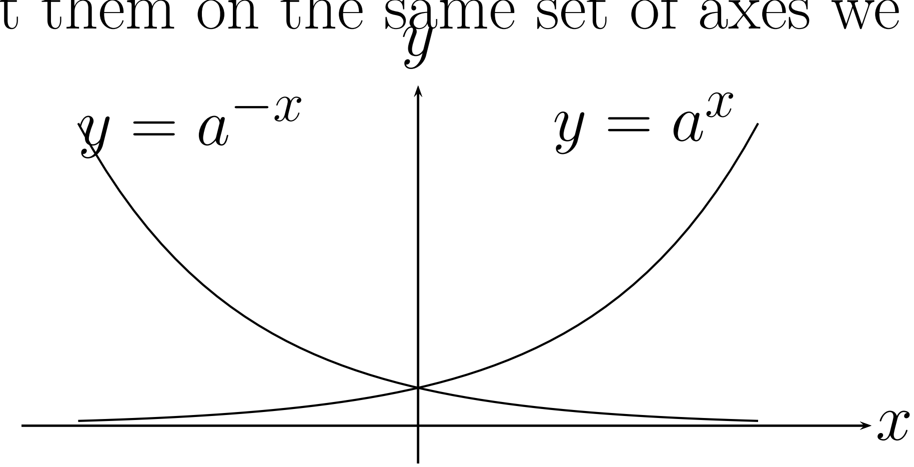
2^x ← No x-value to make negative or zero

What about the graph of $f(x) = 2^{-x}$?

$$2^{-x} = \frac{1}{2^x}$$



In general we have two basic shapes. $y = a^x$ and $y = a^{-x}$. If we plot them on the same set of axes we can see they are very similar:



The Natural Base e

The Number e

The following limits produce the same number and we call that number e.

$$\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x = e$$

$$\lim_{x\to 0}(1+x)^{1/x} = e$$

Not a 1

$e \approx 2.71828\ldots$

$e = 2.718281828\ldots\ldots$

$\boxed{e \text{ is just a number}}$

Since e is a number we can use it in the exponential function $f(x) = e^x$. Why? Because it works in many situations.

$f(x) = e^x$

Properties of exponents

Let a and b be positive numbers with $a \neq 1$, $b \neq 1$ and let x and y be real numbers. Then:

A) Exponent Laws:

$$a^2 \cdot a^3 = \underbrace{a \cdot a \cdot a \cdot a \cdot a}_{5} = a^5 \quad 5 = 2+3$$

1. $a^x a^y = a^{x+y}$
2. $(a^x)^y = a^{xy}$
3. $(ab)^x = a^x b^x$
4. $\left(\frac{a}{b}\right)^x = \frac{a^x}{b^x}$
5. $\frac{a^x}{a^y} = a^{x-y}$

$$(a^2)^3 = a^2 \cdot a^2 \cdot a^2 = a \cdot a \cdot a \cdot a \cdot a \cdot a = a^6$$

$$(a \cdot b)^3 = a \cdot b \cdot a \cdot b \cdot a \cdot b = a \cdot a \cdot a \cdot b \cdot b \cdot b = a^3 b^3$$

$$\frac{a^x}{a^y} = a^x \cdot a^{-y} = a^{x-y}$$

B) $a^x = a^y$ if and only if $x = y$.

$$a^0 = 1 \qquad 1 = \frac{a^n}{a^n} = a^{n-n} = a^0$$

6.1.2 Logarithms

Let $a > 0$, $a \neq 1$ and $x > 0$ then

base a

Exponent

$$y = \log_a(x) \iff x = a^y$$

The function $f(x) = \log_a(x)$ is called the **logarithmic function with base a**

$y = \log_b x \iff x = b^y$ ← exponent

OR

Exponent

"log base a"

Example 6.1.1.

If and only if.

1. $\log_3(81) = 4 \Leftrightarrow 3^4 = 81$

IS

$\log_3(81)$

2. $\log_{16} 8 = \frac{3}{4} \Leftrightarrow 16^{3/4} = 8$

3. $10^{-3} = 0.001 \Leftrightarrow \log_{10}(0.001) = -3$

Base

$$\frac{10^{-3}}{1} \Rightarrow \frac{1}{10^3} = \frac{1}{1000}$$

Example 6.1.2. Find y. $\log_5(125) = y$ Exponent

$$5^y = 125 = 5^3$$

$$y = 3$$

In words: The logarithm $y = \log_a(x)$ is the power (y) to which the base (a) must be raised to get a given number (x)

Example 6.1.3. Rewrite as an exponential function
$\log_2(1/32) = -5$

$$2^{-5} = \frac{1}{32}$$

$$(\log_5 5)x$$

Example 6.1.4. Rewrite as an exponential function
$\log_{16}(2) = 1/4$

$$16^{1/4} = 2$$

$$\log_5 5x = \log_5(5x)$$

$$\log_5(5+x)$$

Example 6.1.5. Rewrite as a logarithmic function $125 = 5^3$

$$\boxed{\log_5(125)} = 3$$

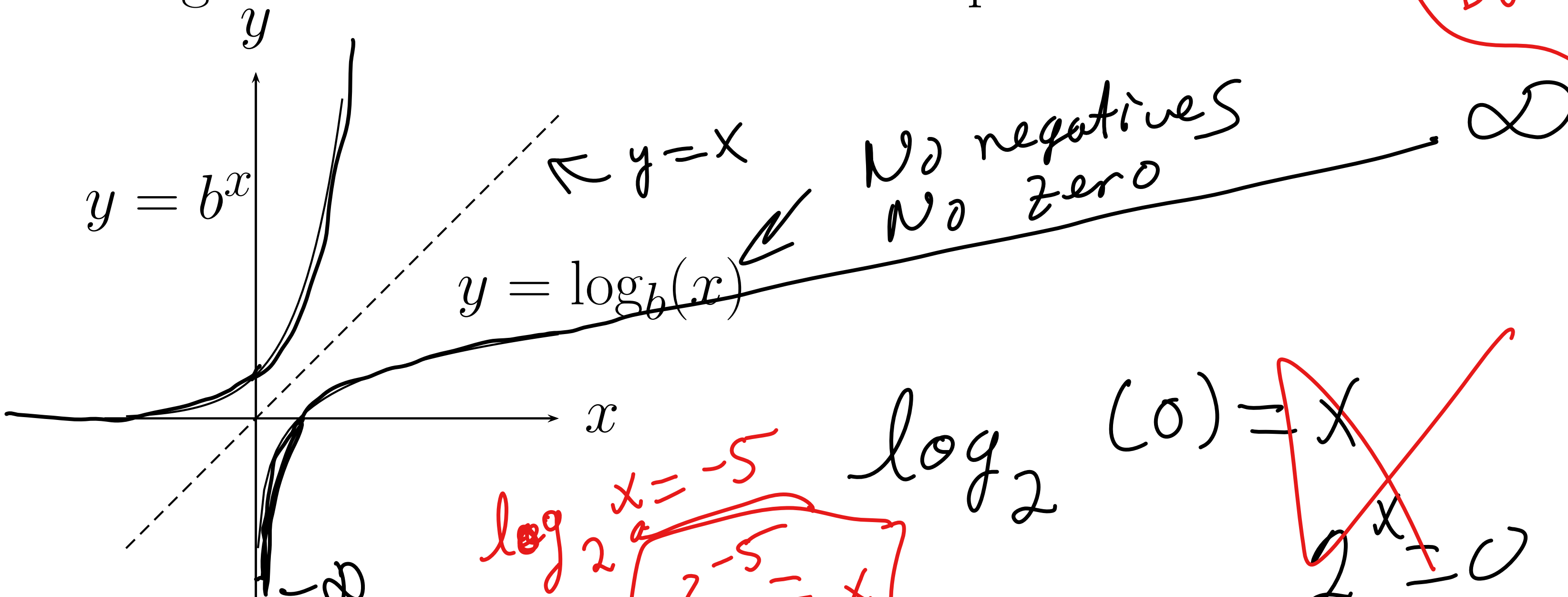
$$\log_5 125 = 3$$

6.1.3 Graphing logarithms

Suppose we have the function $y = f(x) = b^x$.

Then the inverse function is $x = b^y$ OR $y = f^{-1}(x) = \log_b(x)$. So the logarithm is the inverse of the exponential function.

Domain for $\log_b(x)$ is $x > 0$



$\log_2 x = -5$

$\boxed{2^{-5} = x}$

$\log_2(0) = x$

$\cancel{2^x = 0}$

$\log_2(-5) = x$

$\cancel{2^x = -5}$

Since the two functions are inverses of each other then

$$b^{\log_b(x)} = x \qquad \text{AND} \qquad \log_b(b^x) = x$$

$b^x = b^x$

Example 6.1.6. Simplify $\log_2(64)$ $= x$

$2^x = 64 = 2^6$ $\boxed{x=6}$

2	$= 2^1$
4	2^2
8	2^3
16	2^4
32	2^5
64	2^6

Properties of Logarithms

Let b be a positive real number with $b \neq 1$, and let x be any real number. Then:

1. $\log_b(1) = 0$ i.e. $b^0 = 1$
2. $\log_b(b) = 1$ i.e. $b^1 = b$
3. $\log_b(b^x) = x$ i.e. $b^x = b^x$
4. $b^{\log_b(x)} = x$ if $x > 0$

Inverse properties.

6.1.4 The natural logarithm

This is the same as before but now we use base e where e is the number we found in section 6.1.1. Since the log base e shows up so often we call it the **natural log**.

$$\log_e(x) = \ln(x)$$

We also use log base 10 very often so we abbreviate that as

$$\log_{10}(x) = \log(x).$$

Your calculator follows the same convention.

Example 6.1.7. Evaluate the following logarithms without a calculator

1. $\log(1000) = \log_{10}(1000) = x$

$$10^x = 1000 \qquad x = 3$$

2. $\log_9(243) = x$ $\quad (3^2)^x = 9^x = 243 = 3^5$

$3^{2x} = 3^5 \implies 2x = 5$

$\boxed{x = 5/2}$

3. $\underbrace{\log_b(b^{-3})} = \boxed{-3}$

4. $\underbrace{\ln(e^{-2})} = \log_e(e^{-2}) = \boxed{-2}$

Example 6.1.8. Evaluate with your calculator:

1. $\log(345) = 2.537819095$

2. $\log(4/5) = -0.096910013$

3. $3\ln(1+\sqrt{3}) = 3.015157616$

$3 * ((1 + 3(\text{sqrt})) \ln)$

$1 + 3\sqrt{\ } = \text{ans}$

$\ln$

$* 3$

Example 6.1.9. More problems

1. Rewrite $\log_{64} 8 = \frac{1}{2}$ in exponential form.

2. Rewrite $4^{-2} = \frac{1}{16}$ in logarithmic form.

3. Evaluate $\log_8 2$.

4. Use the properties to evaluate $\ln\left(\frac{1}{e^{21}}\right)$

5. Find the domain of $g(x) = \ln(13 - x)$

6. Use the properties to evaluate $-73\ln(e)$

7. Use the properties to evaluate $e^{\ln(55)}$

6.2 Properties of Logarithms

$b^x = y \iff \log_b y = x$

exponent

Change of Base Formula

Let a, b, x be positive real numbers with $a \neq 1, b \neq 1$. Then

$$\log_a(x) = \frac{\log_b(x)}{\log_b(a)} \quad \text{(For any } b\text{)}$$

For the calculator you can use either base 10 or base e.

$$\log_a(x) = \frac{\log(x)}{\log(a)} \qquad \text{OR} \qquad \log_a(x) = \frac{\ln(x)}{\ln(a)}.$$

base 10 base e

Example 6.2.1. Evaluate on a calculator using both common and natural logs.

$$\log_7(4) = \frac{\log 4}{\log 7} = \frac{\ln 4}{\ln 7} = 0.712414374$$

Example 6.2.2. Convert $\frac{60(1.08)^{t}}{60} = \frac{3360}{60}$ to logarithmic form and solve for t

this is a single number

$1.08^t = 56$ write as log $\iff$ $t = \boxed{\log_{1.08}(56)} = \frac{\log 56}{\log(1.08)} = \boxed{52.3}$

Important Properties of Logarithms

Let b, M, N be positive real numbers with $b \neq 1$, and let p be any real number. Then:

1. $\log_b(MN) = \log_b(M) + \log_b(N)$
2. $\log_b\left(\frac{M}{N}\right) = \log_b(M) - \log_b(N)$
3. $\log_b(M^p) = p\log_b(M)$
4. $\log_b(M) = \log_b(N) \iff M = N$

Memorize these

$\underbrace{\log(2\cdot 5)}_{\log 10 = 1} = \log(2) + \log(5) = 1$

$\log(12^5) = 5\log(12)$

Example 6.2.3. Use the properties of logs to rewrite the following expressions.

mult. add

1. $\log_{10}(10z) = \log_{10}(10) + \log_{10}(z) = \boxed{1 + \log_{10} z}$

$\log \frac{M}{N} = \log M - \log N$

2. $\log_6\left(\frac{y}{2}\right) = \boxed{\log_6 y - \log_6 2}$

$\log(M^p) = p\log(M)$

3. $\log_5(\sqrt[3]{5}) = \log_5\left(5^{1/3}\right) = \frac{1}{3}\log_5(5) = \boxed{\frac{1}{3}}$

$$4.\ \ln\left(\frac{x^2-1}{x^3}\right) = \ln\left(\frac{(x-1)(x+1)}{x^3}\right)$$

$$= \ln[(x-1)(x+1)] - \ln(x^3)$$

$$\boxed{= \ln(x-1) + \ln(x+1) - 3\ln x}$$

$$\ln\frac{M}{N} = \ln M - \ln N$$

$$\ln MN = \ln M + \ln N$$

$$\ln M^p = p \ln M$$

$$y^4\sqrt{x}$$

$$5.\ \log\left(\frac{\sqrt{x}y^4}{z^4}\right) = \log\left(\frac{x^{1/2}y^4}{z^4}\right)$$

$$= \log(x^{1/2}y^4) - \log(z^4)$$

$$= \log x^{1/2} + \log y^4 - \log z^4$$

$$\boxed{= \tfrac{1}{2}\log x + 4\log y - 4\log z}$$

Example 6.2.4. Use the properties of logs to rewrite the following expressions as a single logarithm.
~~logarithm.~~ $\ln\left(\frac{M}{N}\right) = \ln M - \ln N$

1. $\ln(x-2) - \ln(x+2) = \boxed{\ln\left(\frac{x-2}{x+2}\right)}$

2. $\underbrace{4\log_b w + 3\log_b x - \log_b y}$ $= \log_b(w^4 x^3) - \log_b y = \boxed{\log_b\left(\frac{w^4x^3}{y}\right)}$

3. $4[\ln z + \ln(z+5)] - 2\ln(z-5) = 4\ln[z(z+5)] - 2\ln(z-5)$

$= \ln[z(z+5)]^4 - \ln(z-5)^2 = \boxed{\ln\left[\frac{[z(z+5)]^4}{(z-5)^2}\right]}$

$= 4\ln z + 4\ln(z+5) - 2\ln(z-5) = \boxed{\ln\left(\frac{z^4(z+5)^4}{(z-5)^2}\right)}$

Example 6.2.5. Solve without a calculator: $\log_4 2 + \log_4 32 =$

$\log_4 2 + \log_4 32 = \log_4(2\cdot 32) = \underbrace{\log_4(64) = x}$

$4^x = 64 = 4^3$

$\boxed{x=3}$

Example 6.2.6. More examples

1. Simplify by using common logarithms: $\log_2(29) = \dfrac{\log(29)}{\log 2} =$

$$\boxed{= 4.857980995}$$

2. Simplify by using natural logarithms: $\log_4 7 = \dfrac{\ln(7)}{\ln(4)} =$

$$= 1.403677461$$

3. Write the expression in terms of $\ln 5$ and $\ln 3$: $\ln \dfrac{5}{81}$

$$\ln\left(\frac{5}{81}\right) = \ln\left(\frac{5}{3^4}\right) = \ln 5 - \ln(3^4) = \boxed{\ln 5 - 4\ln 3}$$

4. Expand the logarithmic expression: $\ln \dfrac{\sqrt[7]{x-1}}{2} = \ln \dfrac{(x-1)^{1/7}}{2}$

$$\boxed{= \frac{1}{7}\ln(x-1) - \ln 2}$$

5. Use the properties of logarithms to expand the expression:

$\log_6(12xy^3) = \log_6 12 + \log_6 x + \log_6 y^3$

$\boxed{= \log_6 12 + \log_6 x + 3\log_6 y}$

$\log M + \log N = \log(MN)$

6. Write as a single logarithm: $\frac{1}{3}[\log_6 x + \log_6(x+7)]$

$= \frac{1}{3}\log_6[x(x+7)] = \boxed{\log_6[x(x+7)]^{1/3}}$

Distribute

7. Write as a single logarithm: $\frac{1}{8}[\log_8(x-4) + 5\log_8(x^2+9)]$

$\frac{1}{8}\log_8(x-4) + \frac{5}{8}\log_8(x^2+9)$

$= \log_8(x-4)^{1/8} + \log_8(x^2+9)^{5/8} = \boxed{\log_8\left[(x-4)^{1/8}\cdot(x^2+9)^{5/8}\right]}$

If $\ln a = 2$, $\ln b = 3$, and $\ln c = 5$, evaluate the following:

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Section 6.2 #13

(d) $(\ln c^3)\left(\ln \frac{a}{b^{-1}}\right)^{-4} =$

outside Logarithm

inside Logarithm

$$= \frac{\ln c^3}{\left(\ln\left(\frac{a}{b^{-1}}\right)\right)^4}$$

algebra

$$= \frac{3 \ln c}{\left[\ln a - \ln(b^{-1})\right]^4}$$

log properties

$$= \frac{3 \ln c}{\left[\ln a + \ln b\right]^4}$$

$$= \frac{3(5)}{[2+3]^4} = \boxed{\frac{15}{625}}$$

6.3 Exponential Equations

Recall:

$$a^x = a^y \iff x = y$$

and

$$\log_a x = \log_a y \iff x = y$$

Example 6.3.1. Solve for x in the following equations:

1. $64^{x+3} = 16$

$(4^3)^{x+3} = 4^2$

$4^{3(x+3)} = 4^2$

$3(x+3) = 2$

$3x + 9 = 2$

$3x = -7$

$\boxed{x = -7/3}$

$64 = 4^3 = 2^6$

$16 = 4^2 = 2^4$

2. $3^x = 243 = 3^5$ $\boxed{x = 5}$

$\frac{1}{4} = 4^{-1}$

3. $\left(\frac{1}{4}\right)^x = 64 \Rightarrow 4^{-x} = 4^3 \Rightarrow \boxed{x = -3}$

4. $2^{x-3} = 32 = 2^5$

$x - 3 = 5$

$\boxed{x = 8}$

$\ln x = \log_e x$

$\longrightarrow \log_b (b^x) = x$

5. $e^{9x} = 15$ ← take ln on both sides.

$\ln(e^{9x}) = \ln(15)$

$9x = \ln 15$

$\boxed{x = \frac{\ln 15}{9}}$

$e^{9\left(\frac{\ln 15}{9}\right)} = e^{\ln 15} = 15$

6. $2 + e^{x+2} = 32$

$\ln(e^{x+2}) = \ln(30)$

$x + 2 = \ln 30$

$\boxed{x = \ln 30 - 2}$

$\log(e^{x+2}) = \log(30)$

$\frac{(x+2)\log e}{\log e} = \frac{\log(30)}{\log e}$

$\boxed{x \cancel{+2} = \frac{\log 30}{\log e} - 2}$

Example 6.3.2. Solve for x:

1. $e^{2x} = 50$

$$\ln(e^{2x}) = \ln(50)$$

$$\frac{2x}{2} = \frac{\ln 50}{2}$$

$$\boxed{x = \frac{\ln 50}{2}}$$

2. $6^{5x} = 3000$

$$\ln(6^{5x}) = \ln(3000)$$

$$\frac{5x \ln 6}{5 \ln 6} = \frac{\ln(3000)}{5 \ln 6}$$

$$\boxed{x = \frac{\ln(3000)}{5 \ln 6}}$$

3. $\dfrac{119}{e^{6x} - 14} = 7$

$$17 = \frac{119}{7} = \frac{7(e^{6x} - 14)}{7}$$

$$17 = e^{6x} - 14$$

$$+14 \qquad\qquad +14$$

$$\ln 31 = \ln(e^{6x})$$

$$\frac{\ln 31}{6} = \frac{6x}{6}$$

$$\boxed{x = \frac{\ln 31}{6}}$$

6.4 Logarithmic Equations

$\log M^p = p \cdot \log M$

Example 6.4.1. Which of ~~the following~~ these are equivalent to $3 \log\left(u^{-1/7}\right)$?

1) $\log\left(\dfrac{1}{\sqrt[7]{u^3}}\right) = \log\left(\dfrac{1}{u^{3/7}}\right) = \log\left(u^{-3/7}\right) = \log\left(\left(u^{-1/7}\right)^3\right) = 3\log u^{-1/7}$

2) $-\dfrac{3}{7}\log(u) = 3\left(-\dfrac{1}{7}\right)\log u = \boxed{3\log u^{-1/7}}$

3) $\log\left(\dfrac{1}{u^{3/7}}\right)$ Done

4) $-\dfrac{1}{7}\log\left(u^3\right) = 3\left(-\dfrac{1}{7}\right)\log u = 3\log\left(u^{-1/7}\right)$

5) $-\log\left(u^{3/7}\right) = 3\log\left(u^{-1/7}\right)$

$\log_b x = y \iff b^y = x$

Example 6.4.2. Solve for x in the following equations:

1. $\underbrace{\ln x - \ln 5}_{} = 0$ $\qquad \ln x = \ln 5 \qquad \log M - \log N = \log \frac{M}{N}$

$$\ln \frac{x}{5} = 0 \iff e^0 = \frac{x}{5} \qquad \ln x = \log_e x$$

$$\boxed{x = 5}$$

write as exponential

2. $\log_x 625 = 4$

$$x^4 = 625 \qquad \boxed{x = 5}$$

3. $\ln(2x - 1) = 0$

$$1 = e^0 = 2x - 1$$

$$2x - 1 = 1$$
$$2x = 2$$
$$\boxed{x = 1}$$

4. $5 + 5 \ln x = 30$

$$5 \ln x = 25$$
$$\ln x = 5$$

write as exponential

$$\boxed{e^5 = x}$$

5. $\log 2x - \log 8x^2 = 3$

Logarithmic equations

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$\log M - \log N = \log \frac{M}{N}$

$$5.\ \underbrace{\log 2x - \log 8x^2} = 3$$

$$\log \frac{2x}{8x^2} = 3$$

$$\log \frac{1}{4x} = 3$$

write as exponential

$$10^3 = \frac{1}{4x}$$

$$x = \frac{1}{4(10^3)} = \boxed{\frac{1}{4000} = x}$$

Example 6.4.3. Simplify the following:

1. $\log_6 6^{2x-1} = (2x-1)\underbrace{\log_6 6}_{1} = \boxed{2x-1}$

2. $\ln e^{x^4} = x^4$

Example 6.4.4. Solve for x:

1) $\ln 4x = 1$

$e^1 = 4x$

write as exponential

$\boxed{\frac{e}{4} = x}$

2) $\log_6 6x + \log_6(x+5) = 2$

$$\log M + \log N = \log(MN)$$

2) $\log_6 6x + \log_6(x+5) = 2$

$$\log_6 (6x(x+5)) = 2$$

$$6^2 = 6x(x+5)$$

$$36 = 6x^2 + 30x$$

$$(0 = 6x^2 + 30x - 36)\frac{1}{6}$$

$$0 = x^2 + 5x - 6$$

$$0 = (x+6)(x-1)$$

$$\cancel{x = -6}, \quad x = 1$$

$$\log_6(6x) + \log_6(x+5) = 2$$

$x = -6$ $\quad \log_6(6(-6))$ can't be done

$x = 1$ $\quad \log_6 6 + \log_6 6 = 1 + 1 = \boxed{2}$

3) $\log_6 x^2 - \log_6(x+1) = 2$

$$\log_6\left(\frac{x^2}{x+1}\right) = 2$$

write as exponential

$$36 = 6^2 = \frac{x^2}{x+1}$$

$$36(x+1) = x^2$$

$a=1$, $b=-36$, $c=-36$

$$0 = x^2 - 36x - 36$$

$$x = \frac{-b \pm \sqrt{b^2-4ac}}{2a} = \frac{36 \pm \sqrt{36^2 - 4(1)(-36)}}{2(1)}$$

check in original

$$x = \frac{36 \pm \sqrt{1440}}{2}$$

≈ -0.973665961 ✓

≈ 36.97366596 ✓

$1P + rP = P(1+r)$

6.5 Exponential and Logarithmic Models

$2x + r \cdot x = x(2+r)$

Finance: Compound interest.

$I = P \cdot r \cdot t$

P is some principal amount of money. 100

r is the interest rate per year. $5\% = 0.05$

t – time

Simple interest:

$100 + (0.05)100 = 105 = 100(1+0.05)$

$t=1$

After 1 year we have $P_1 = P + rP = P(1+r)$

$t=2$

After 2 year we have $P_2 = P_1 + rP_1 = P_1(1+r) = P(1+r)(1+r) = P(1+r)^2$

Interest

$= P(1+r)^2$

In fact at year t you have

$$P_t = P(1+r)^t$$

← exponential equation

If you compound each month then you have to add 1/12 of the interest every month and you get

$$P_t = P\left(1 + \frac{r}{12}\right)^{12t}, \qquad \text{where } t = \text{number of years.}$$

In general if you compound n times per year you have

$$P_t = P\left(1 + \frac{r}{n}\right)^{nt}.$$

If we compound continuously that mean $n \to \infty$ so

$$P_t = Pe^{rt}.$$

6.5.1 6.5.3 6.5.2

Example 6.5.1. My credit card has \$4000 on it. The interest rate is 18% per year. If I make no payments how much do I owe at the end of three years if the credit card company compounds the interest (a) yearly, (b) monthly, (c) daily, and (d) continuously?

$t=3$

$P_t = Pe^{rt}$

$P_t = P(1+\frac{r}{n})^{nt}$ (a) (b) (c)

$r = 0.18$, $t = 3$, $P = 4000$

$n = 1$ for (a) $n = 365$ for (c)

$n = 12$ for (b) $n = \infty$ for (d)

a) $n = 1$ $\quad P_3 = P\left(1 + \frac{r}{n}\right)^{nt}$

$= 4000\left(1 + \frac{0.18}{1}\right)^{1(3)} = \boxed{6572.13}$

b) $n = 12$ $\quad P_3 = P\left(1 + \frac{r}{n}\right)^{nt}$

$= 4000\left(1 + \frac{0.18}{12}\right)^{12(3)} = \boxed{6836.56}$

c) $n = 365$ $\quad P_3 = 4000\left(1 + \frac{0.18}{365}\right)^{365(3)} = \boxed{6863.11}$

d) continuous $\quad P_3 = Pe^{rt}$

$= 4000\, e^{(0.18)(3)} = \boxed{6864.03}$

Example 6.5.2. We would like to have \$50000 in our investement account at the end of t years. Find the amount of principal P that must be invested at 9.5% compounded continuously if

$P_t = 50{,}000$

$P_t = P e^{rt}$

Don't now this

$P = \frac{P_t}{e^{rt}}$

1. $t = 1$ year

$5000 = P e^{(0.095 \cdot 1)}$

$P = \frac{50{,}000}{e^{0.095}} = \boxed{\$45{,}468.65}$

2. $t = 10$ years

$P = \frac{P_t}{e^{rt}} = \frac{50{,}000}{e^{(0.095)(10)}} = \boxed{\$19{,}337.05}$

3. $t = 50$ years.

$P = \frac{50000}{e^{(0.095)(50)}} = \boxed{432.58}$

Example 6.5.3. Find the time required to double an investment of P_0 dollars if the interest rate is 6% compounded continuously.

The formula for interest compounded continuously is

$$P_t = P_0 e^{rt}.$$

$r = 0.06$

To double the investment we want to find the value of t when $P_t =$ ~~$2P_0$.~~

$P_0 =$ Starting amount

$P_t = 2P_0$

$$\frac{2\cancel{P_0}}{\cancel{P_0}} = \frac{\cancel{P_0} e^{rt}}{\cancel{P_0}} \Rightarrow 2 = e^{0.06t}$$

$$\ln(2) = \underbrace{\ln\left(e^{0.06t}\right)}$$

$$\frac{\ln 2}{0.06} = \frac{0.06t}{0.06}$$

$$\boxed{t = 11.55 \text{ years}}$$

\$1000 at 18 years old

2000 at 29

4000 at 41

8000 at 52

16000 at 64

Example 6.5.4. The population of a city is growth rate 0.012 OR 1.2%

$$P = 240360e^{0.012t}$$

where $t = 0$ represents the year 2000. According to this model when will the population reach 275000?

$$\frac{275{,}000}{240{,}360} = \frac{240360\, e^{0.012t}}{240360}$$

Solve for t

step 1: exponential by itself

$$\ln\left(\frac{275000}{240360}\right) = \ln e^{0.012t}$$

step 2: log

$$\frac{\ln\left(\frac{275{,}000}{240{,}360}\right)}{0.012} = \frac{0.012\,t}{0.012}$$

Step 3: divide

$t = 11.2$ years

in the year 2011

Example 6.5.5. The number of bacteria N in a culture is modeled by

Initial

$$N = 250e^{kt}$$

where t is the time in hours. After 10 days the population is 280 bacteria. (ie. $N = 280$ when $t = 10$), estimate the time required to double the population.

step 1: Find k = growth rate

$t = 10$
$N = 280$

$$\frac{280}{250} = \frac{250\, e^{k(10)}}{250}$$

$$\ln \frac{28}{25} = \ln e^{10k}$$

$$\frac{\ln \frac{28}{25}}{10} = \frac{10k}{10}$$

$$\boxed{k = 0.01133}$$

step 2: Find t for $N = 2(250) = 500$. (double the original population)

$$500 = 250\, e^{0.01133\, t}$$

$t = 61$ days (ish)

Example 6.5.6. Carbon 14 (^{14}C) has a half life of 5730 years. (Half life is the amount of time for half the original material to decay.) Carbon 14 dating assumes that the carbon dioxide today has the same amount of radioactive material as it did centuries ago. If this is true, the amount of ^{14}C absorbed by a tree centuries ago should be the same as a tree growing today. A piece of ancient coal has 15% as much ^{14}C as a piece of modern coal. How long ago was the tree burned to make the ancient coal?

modern (today)

Decay Model:

$\ln \frac{1}{2} = \cancel{\ln 1} - \ln 2$

$$A = A_0 e^{-kt}$$

where A_0 is the original amount of material, k is the decay constant and A is the amount of material left after t years.

$k = \frac{\ln 2}{5730}$

Step 1: Find k using the half life. $\frac{1}{2}A_0 = A_0 e^{-k5730}$

$k = -\frac{\ln \frac{1}{2}}{5730}$

Step 2: Find t for $0.15\cancel{A_0} = \cancel{A_0} e^{-kt}$.

$t = -15,683$ years

$$k = \frac{\ln 2}{5730}$$ ← positive

$$\frac{0.15\cancel{A_0}}{\cancel{A_0}} = \frac{\cancel{A_0} e^{-kt}}{\cancel{A_0}}$$ ← negative exponent

$$\ln(0.15) = \ln\left(e^{-kt}\right) = -kt$$

flip
$$\frac{\ln(0.15)}{\frac{\ln 2}{5730}} = \frac{\ln(0.5)}{} = t$$

$$\boxed{t = -15{,}683 \text{ years}}$$

$$\frac{(5730)\ln(0.15)}{\ln 2}$$