5.1 Function Composition

∘ NOT MULTIPLICATION

The **composition** of a function f with a function g is

$$(f \circ g)(x) = f(g(x)).$$

g is in f

The domain of $(f \circ g)$ is the set of all x in the domain of g such that $g(x)$ is in the domain of f.

Saw in Ch2 $f(\underbrace{x+h})$ ← put an expression in a function

Domain for $(f \circ g)(x) = f(g(x))$

Start by looking at domain of $g(x)$.

Example 5.1.1. Suppose $f(x) = x^3 + 2x + 1$ and $g(x) = x - 1$ then find

g into f

a) $(f \circ g)(x) = f(g(x)) = f(x-1) = (x-1)^3 + 2(x-1) + 1$

$$\boxed{= x^3 + 3x^2(-1) + 3x(-1)^2 - 1 + 2x - 2 + 1}$$

$$\begin{array}{ccccc} & & 1 & & \\ & 1 & & 1 & \\ 1 & & 2 & & 1 \end{array}$$

b) $(g \circ f)(x) = g(f(x)) = g(x^3 + 2x + 1)$

$(x+a)^2 = 1x^2 + 2ax + 1a^2$

$(x+a)^3 = 1x^3 + 3x^2a + 3xa^2 + 1a^3$

$(x+a)^4 \quad 1x^4 + 4x^3a + 6x^2a^2 + 4xa^3 + 1a^4$

$= x^3 + 2x + 1 - 1$

$$\boxed{= x^3 + 2x}$$

c) $(f \circ f)(x) = f(f(x)) = f(x^3 + 2x + 1)$

$$\boxed{= (x^3 + 2x + 1)^3 + 2(x^3 + 2x + 1) + 1}$$

Example 5.1.2. Find (a) $(f \circ g)(x)$, (b) $(g \circ f)(x)$ and (c) the domain of each for $f(x) = \sqrt{x-4}$ and $g(x) = x^2$.

a) $(f \circ g)(x) = f(g(x)) = f(x^2) = \boxed{\sqrt{x^2-4}}$

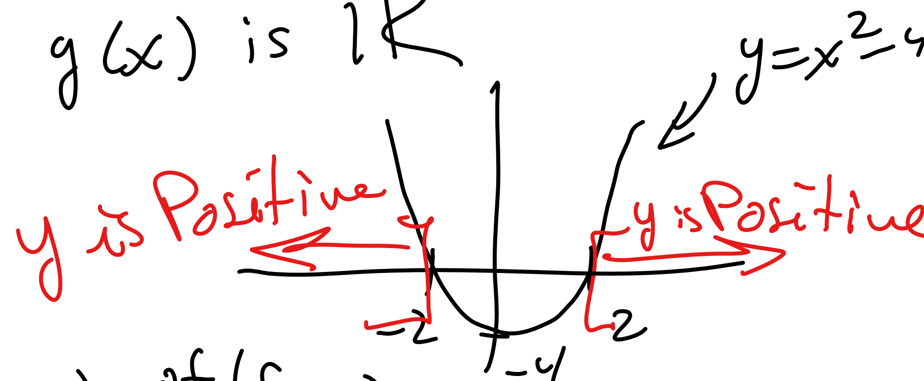
b) $(g \circ f)(x) = g(f(x)) = g(\sqrt{x-4}) = (\sqrt{x-4})^2$

$\boxed{= x-4}$

c) For (a) Domain of $g(x)$ is $\mathbb{R}$

Domain $x^2-4 \geq 0$

$\boxed{(-\infty, -2] \cup [2, \infty)}$ Domain of $(f \circ g)$



For (b) Domain of $f(x)$ is $x-4 \geq 0$

$x \geq 4$

$\boxed{\text{Domain of } (g \circ f)(x) \text{ is } x \geq 4}$

Example 5.1.3. Find (a) $(f \circ g)(x)$, (b) $(g \circ f)(x)$ and (c) the domain of each for

$$f(x) = \frac{1}{x-4} \quad \text{and} \quad g(x) = \frac{2}{x} + 2$$

$x \neq 4$ $\qquad$ $x \neq 0$

(a)

$$(f \circ g)(x) = f(g(x)) = \frac{1}{\left(\frac{2}{x}+2\right) - 4} = \frac{1}{\left(\frac{2}{x} - 2\right)} \cdot \frac{x}{x} = \boxed{\frac{x}{2-2x}}$$

(b)

$$(g \circ f)(x) = g(f(x)) = \frac{2}{x-4} + \frac{2(x-4)}{x-4} = \frac{2+2(x-4)}{x-4} = \boxed{\frac{2x-6}{x-4}}$$

(a) Domain $x \neq 0$ (b/c $g(x)$)

$2 - 2x \neq 0$

$x \neq 1$ (composition)

$\boxed{\mathbb{R} \text{ except } x = 0, 1}$

$\boxed{(-\infty, 0) \cup (0, 1) \cup (1, \infty)}$

(b) domain $x \neq 4$ b/c $f(x)$

& $g(f(x))$

$(-\infty, 4) \cup (4, \infty)$

Example 5.1.4. Find functions f and g such that

$$h(x) = \sqrt[3]{x^2 - 4} = (f \circ g)(x)$$

$f(x) = \sqrt[3]{x}$ $\quad$ $g(x) = x^2-4$ $\quad$ $(f \circ g)(x) = \sqrt[3]{x^2-4}$

$f(x) = \sqrt[3]{x-4}$ $\quad$ $g(x) = x^2$ $\quad$ $(f \circ g)(x) = \sqrt[3]{x^2-4}$

$f(x) = \sqrt[3]{x^2-4}$ $\quad$ $g(x) = x$ $\quad$ $(f \circ g)(x) = \sqrt[3]{x^2-4}$

$f(x) = x$ $\quad$ $g(x) = \sqrt[3]{x^2-4}$

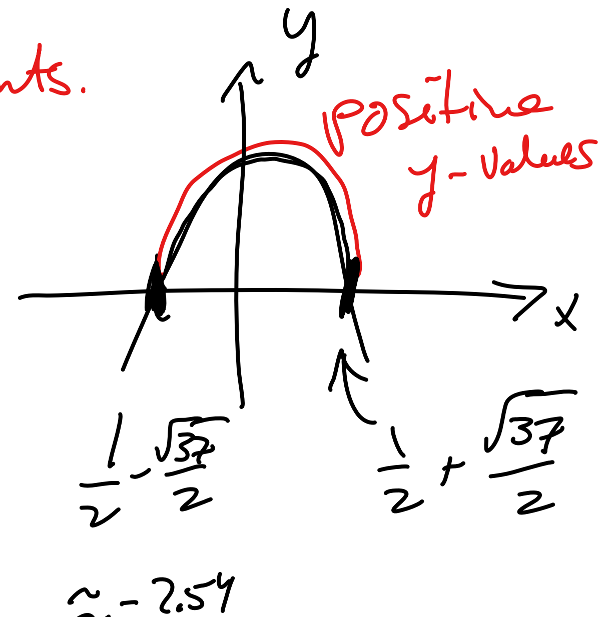
Example 5.1.5. Let $f(x) = \sqrt{90 - x}$ and $\overbrace{g(x) = x^2 - x}^{\text{No restrictions}}$. Find $(f \circ g)(x)$ and its domain.

$$(f \circ g)(x) = f(g(x)) = f(x^2 - x) = \sqrt{9 - (x^2 - x)}$$

$$= \sqrt{9 + x - x^2}$$

Include endpoints.

Domain: $9 + x - x^2 \geq 0$



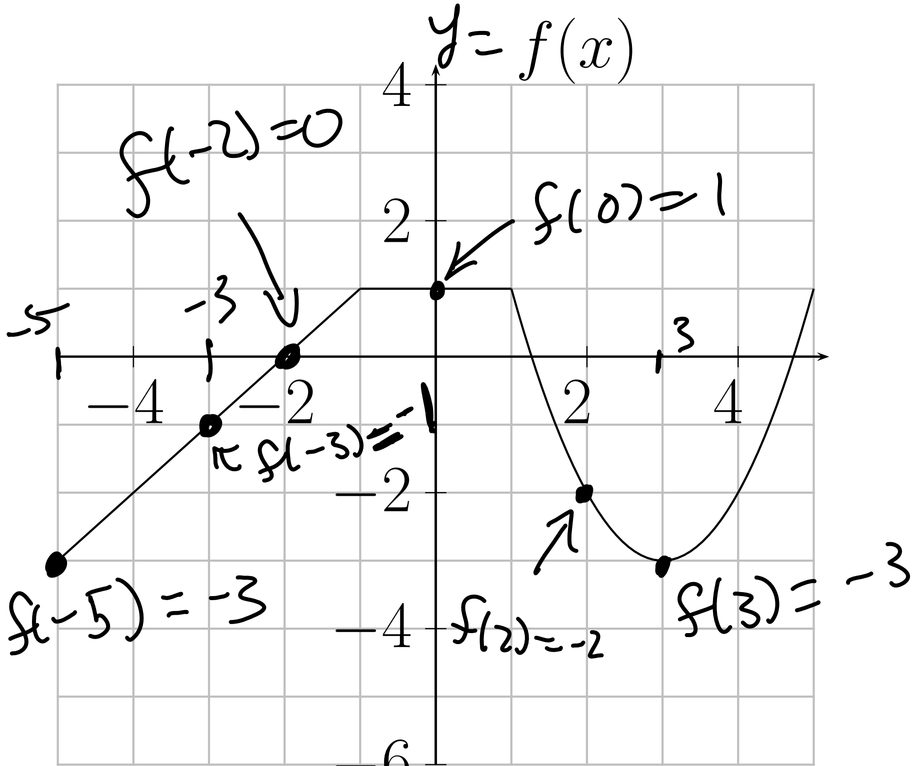
$a = -1$, $b = 1$, $c = 9$

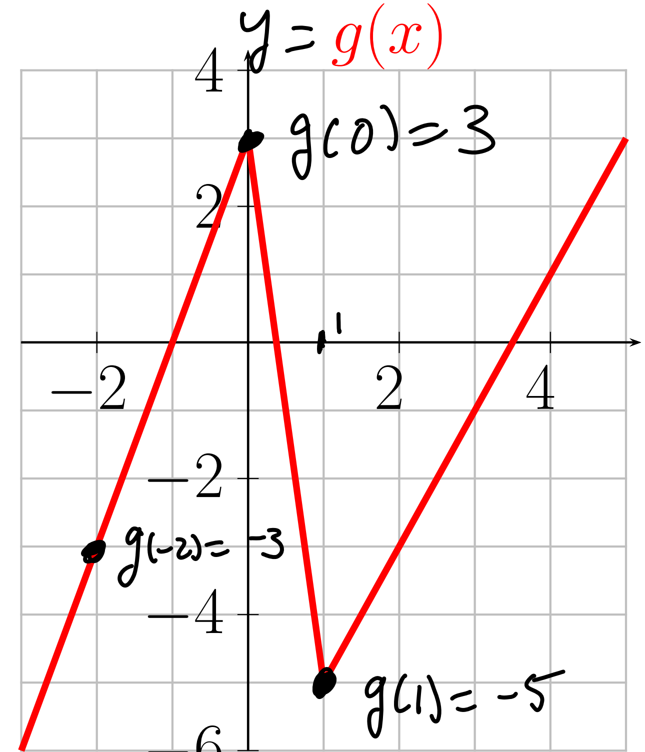
$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-1 \pm \sqrt{1 - 4(9)(-1)}}{2(-1)}$$

$$= \frac{-1 \pm \sqrt{37}}{-2} = \frac{1}{2} \pm \frac{\sqrt{37}}{2}$$

$$\boxed{\left[\frac{1}{2} - \frac{\sqrt{37}}{2}, \frac{1}{2} + \frac{\sqrt{37}}{2}\right]}$$

Example 5.1.6. Use the functions below to find the compositions.





$f(g(1)) = f(-5) = \boxed{-3}$

$g(f(2)) = g(-2) = \boxed{-3}$

$f(g(0)) = f(3) = \boxed{-3}$

$g(f(0)) = g(1) = \boxed{-5}$

$f(g(-2)) = f(-3) = \boxed{-1}$

$g(f(-2)) = g(0) = \boxed{3}$

5.2 Inverse Functions

A function is said to be **one to one (1 - 1)** if no two ordered pairs have the same second component but different first component.

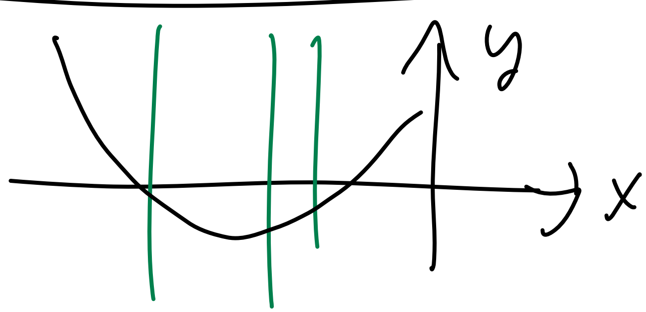
A function has one y value for each x value but those y values can repeat. In a 1 - 1 function the y values never repeat.

$y = x^2$ function with repeat y's

$x = 2 \quad y = 4 \qquad x = -2 \quad y = 4$

$y = x^3$ 1-1 function all y's are unique

all x's & y's are unique



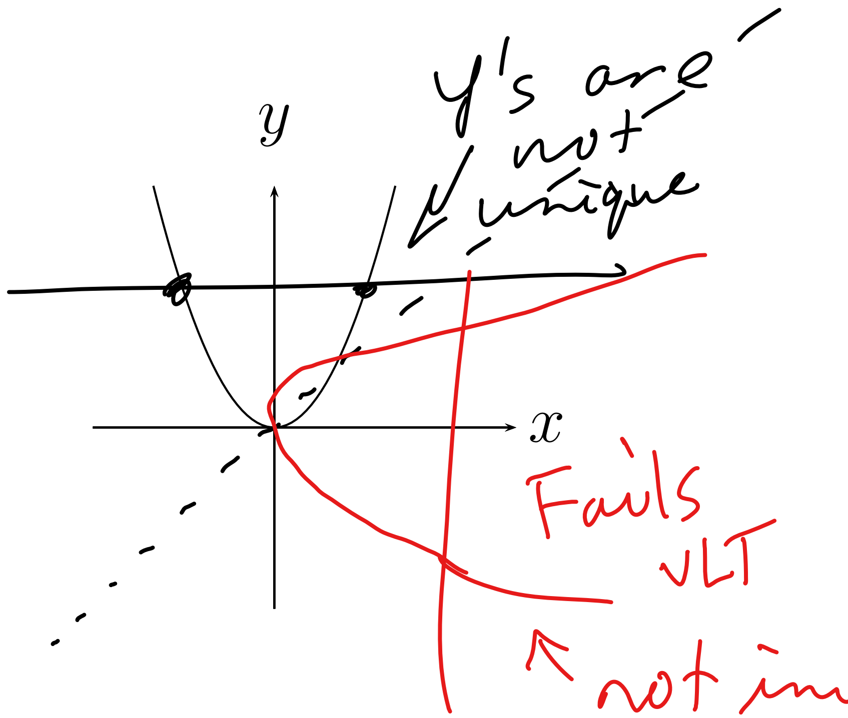
Graphically:

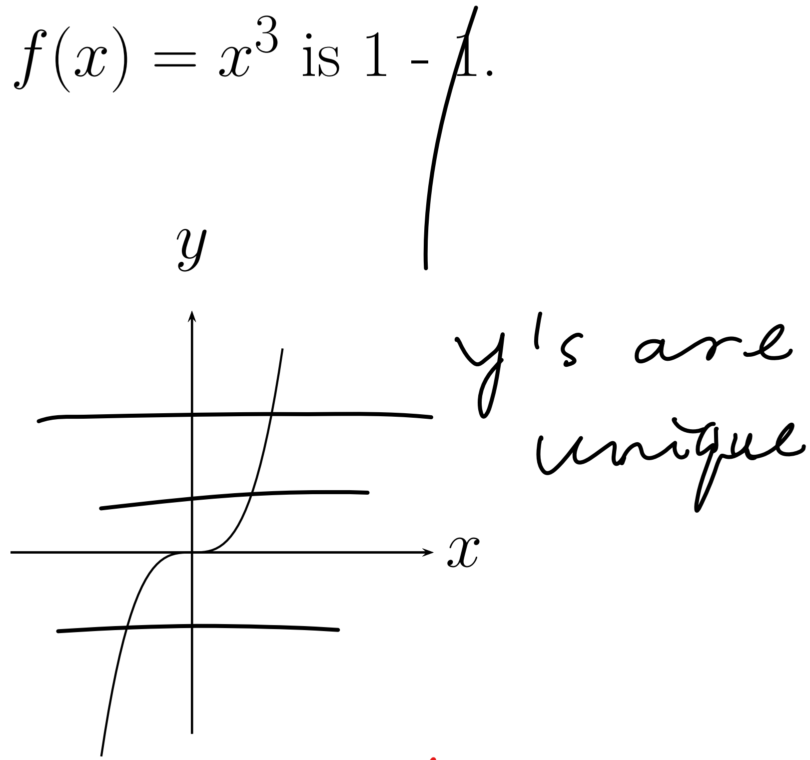
An equation must pass the Vertical Line Test to be a function.

A function must pass the Horizontal Line Test to be 1 - 1.

Example 5.2.1. Use the horizontal line test on these graphs.

$f(x) = x^2$ is not 1 - 1. $\qquad$ $f(x) = x^3$ is 1 - 1.





Inverses

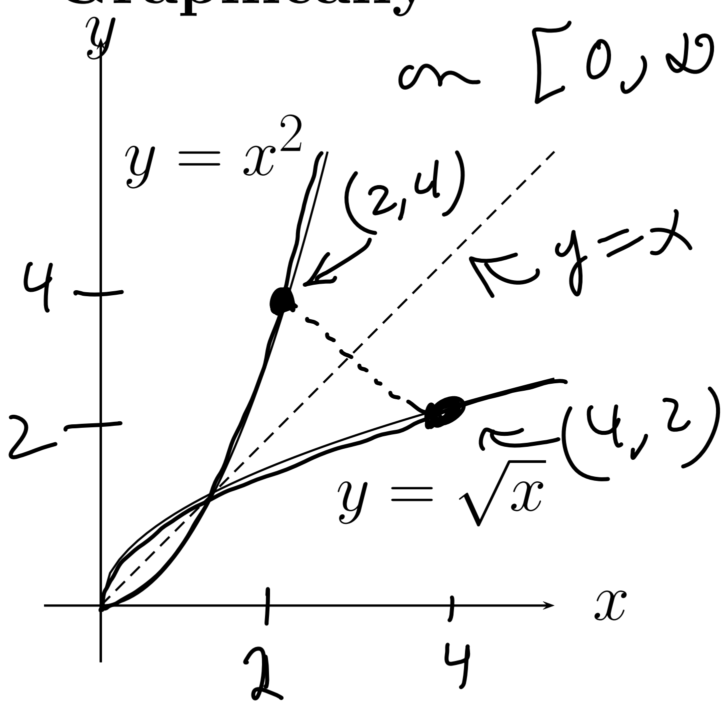
The **indentity function** is $f(x) = x$ or $y = x$. You get out what you put in. Given a function f that is 1 - 1 then f has an inverse f^{-1} and

$$f(f^{-1}(x)) = \boxed{x} \qquad \text{and} \qquad f^{-1}(f(x)) = x.$$

If f is not 1 - 1 then f^{-1} DOES NOT EXIST.

Graphically

on $[0, \infty)$ (Domain restriction)

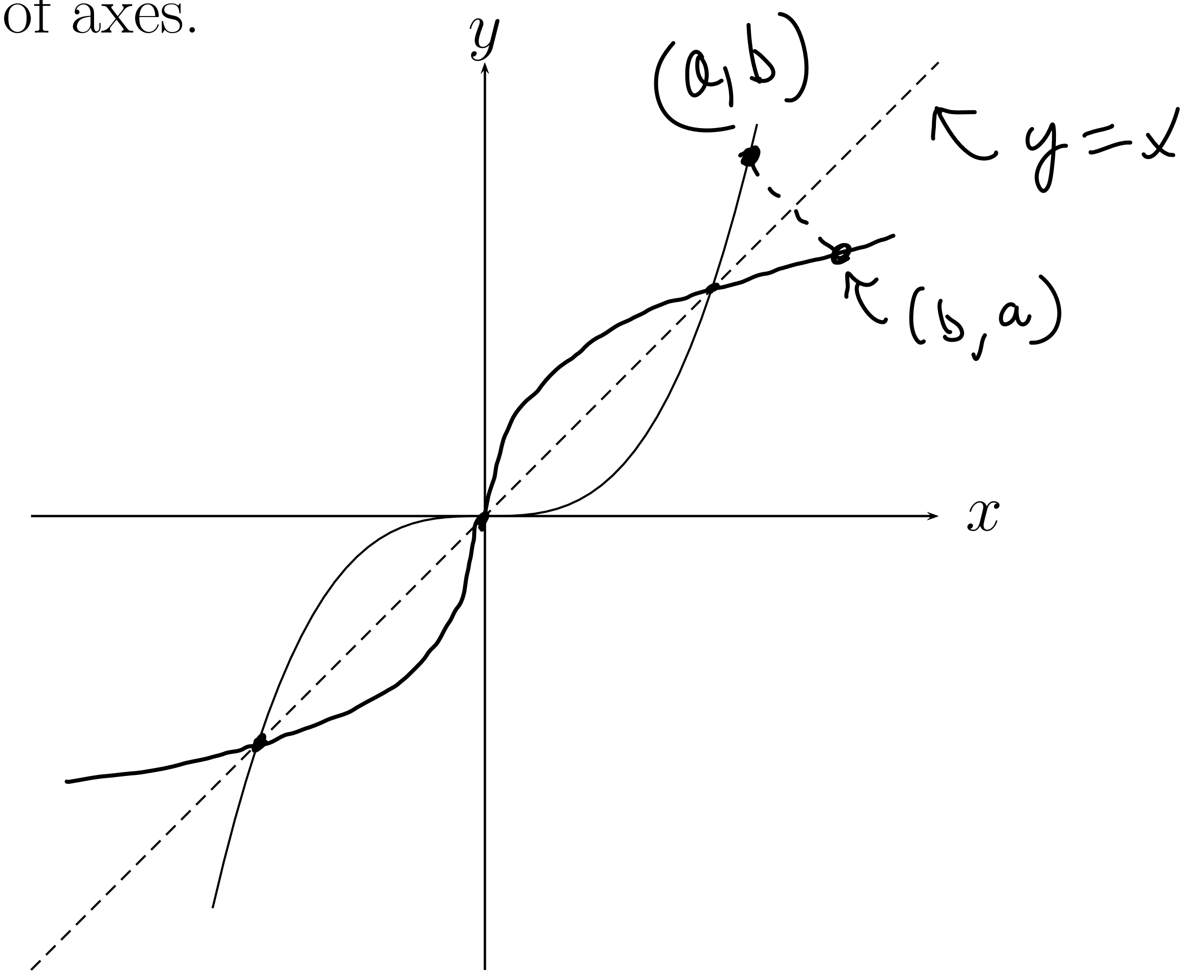


The inverse exchanges
$x \longleftrightarrow y$

$f(x) = x^2$
$g(x) = \sqrt{x}$

$f(g(x)) = f(\sqrt{x}) = (\sqrt{x})^2$
$= x$ Identity

Example 5.2.2. Draw the inverse of the following function on the same set of axes.



5.2.1 Finding inverses from tables and graphs

Example 5.2.3. Use the table below to fill in the missing values.

x	0	1	2	3	4	5	6	7	8	9
$f(x)$	8	4	5	7	2	1	6	9	3	0

$f^{-1}(x) = 1$

$\underbrace{f(f^{-1}(x))}_{x} = f(1)$

$x = f(1) = 4$

1. $\underbrace{f(2)}_{x=2} = 5$ ← y-value

2. if $f(x) = 4$ then $\boxed{x = 1}$

$f(1) = 4$

5 is a y-value

3. $\underbrace{f^{-1}(5)}_{\text{x-value}} = \boxed{2}$ ← x-value

4. if $f^{-1}(x) = 1$ then $x = \boxed{4}$

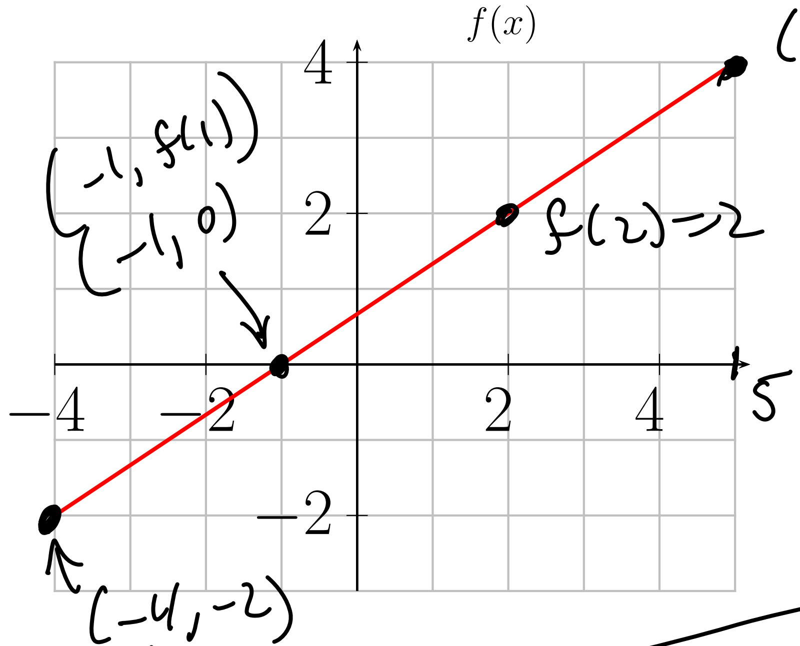
x-value

$x = f(1)$

because $f(1) = 4$

& $f^{-1}(4) = 1$

Example 5.2.4. Use the graph below to fill in the missing values.



2 = x-value

$\underbrace{f(2)}_{\text{y-value}} = 2$

$f(5) = 4$

if $f(x) = 4$ then $x = 5$

$f^{-1}(-2) = -4$

y-value

$f^{-1}(0) = \boxed{-1}$

x-value for y-value = 0

a y-value on our graph

if $f^{-1}(x) = 5$ then $x =$

$x = \underbrace{f(5)} = 4$

$m = \frac{2}{3}$ $(-1, 0)$

$y - y_1 = m(x - x_1)$

$y - 0 = \frac{2}{3}(x + 1)$

$y = \frac{2}{3}x + \frac{2}{3}$

To find inverse switch $x \leftrightarrow y$, Solve for y.

$3(x) = \left(\frac{2}{3}y + \frac{2}{3}\right)3$

$3x = 2y + 2$

$\frac{3x - 2}{2} = \frac{2y}{2}$

$f^{-1}(0) = \frac{-2}{2} = -1$

$\boxed{y = \frac{3x-2}{2} = f^{-1}(x)}$

5.2.2 Finding Inverses Algebraically

OR

Step 1: Solve for x.

1: switch $x \Leftrightarrow y$

Step 2: Check the domain.

2: solve for y

Step 3: Switch x and y.

3: check domain

Step 4: Write $f^{-1}(x) =$

4:

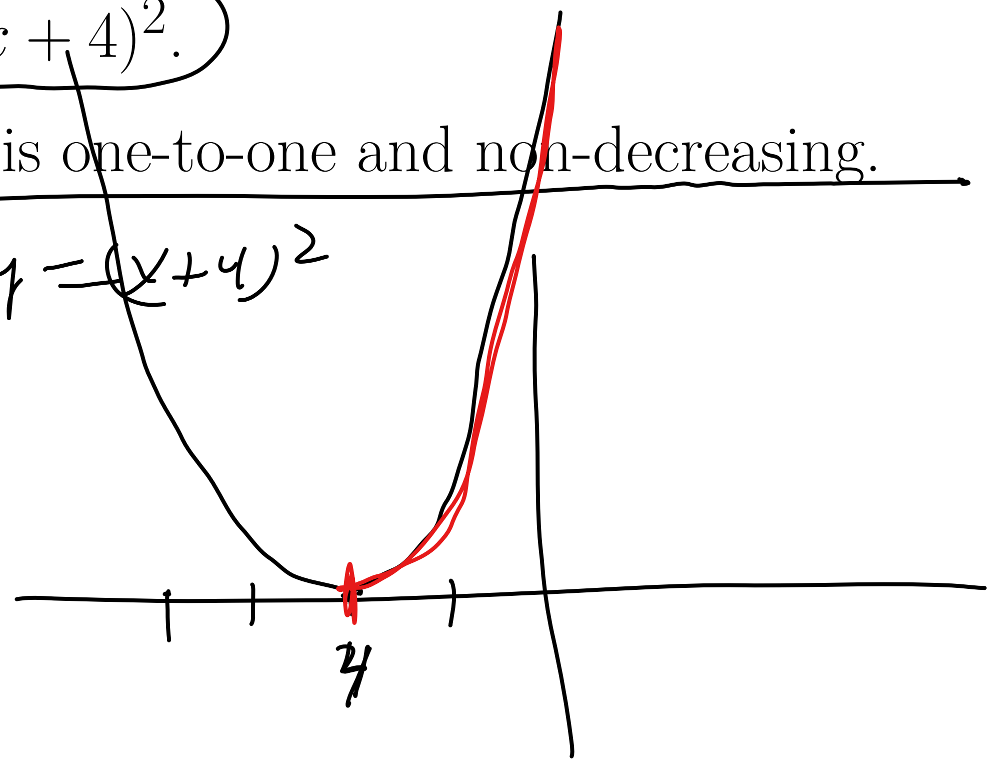
Step 5: Check that $f(f^{-1}(x)) = x$.

5:

Example 5.2.5. Let $f(x) = (x+4)^2$.

1. Find the domain on which f is one-to-one and non-decreasing.

$\boxed{x \geq -4}$



$\pm\sqrt{y}$

2. ~~Find~~ the inverse of $\cancel{f(x)} = \sqrt{(x+4)^2}$ restricted to this domain.

Step 1: Solve for x. (two answers here)

$$\pm\sqrt{y} = x+4 \implies x = -4 \pm \sqrt{y}$$

Step 2: Check the domain.

$x \geq -4$ so choose $x = -4 + \sqrt{y}$

Step 3: Switch x and y.

$$y = -4 + \sqrt{x}$$

Step 4: Write $f^{-1}(x) = -4 + \sqrt{x}$

Step 5: Check that $f(f^{-1}(x)) = x$.

$f(x) = (x+4)^2$ $\quad$ $f^{-1}(x) = -4 + \sqrt{x}$

$$f(f^{-1}(x)) = f(-4+\sqrt{x}) = (-4+\sqrt{x}+4)^2 = (\sqrt{x})^2 = x$$

Solve for x

Example 5.2.6. Find the inverse of $\cancel{f(x)}\ y = \dfrac{2x-5}{-4x-2}$.

$$\left(y = \frac{2x-5}{-4x-2}\right)(-4x-2)$$

Domain $-4x-2 \neq 0$

$-4x \neq 2$

$x \neq -\frac{1}{2}$

$$y(-4x-2) = 2x-5$$

$$\underbrace{-4xy}\ -2y = 2x-5$$

$$-4xy - 2x = 2y-5$$

Factor x

$$x(-4y-2) = 2y-5$$

$$x = \frac{2y-5}{-4y-2}$$

Switch $x \leftrightarrow y$

$$y = \boxed{\frac{2x-5}{-4x-2} = f^{-1}(x)}$$