4.1 Introduction to Rational Functions

Q: What is a rational function?

A: It is a function of the form:

$$f(x) = \frac{N(x)}{D(x)} \qquad D(x) \neq 0.$$

Vertical and Horizontal Asymptotes

A **Vertical Asymptote** describes the behavior of a function near a discontinuity. They occur at any x - value where the numerator IS NOT equal to zero but the denominator IS equal to zero.

Example 4.1.1. Find the domain and vertical asymptotes for

$$f(x) = \frac{1}{x} \quad \text{and} \quad f(x) = \frac{1}{x-3} \quad \text{and} \quad f(x) = \frac{1-3x}{x^2+12x+32}.$$

A **Horizontal Asymptote** describes the behavior of a function as x gets very large. (ie. What happens to y as x goes to ∞?)

Horizontal Asymptotes

$$f(x) = \frac{N(x)}{D(x)} = \frac{a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0}{b_m x^m + b_{m-1}x^{m-1} + \cdots + b_1 x + b_0}$$

where $N(x)$ and $D(x)$ have no common factors. The graph of f has one or no **horizontal asymptote** determined by comparing the degrees of $n(x)$ and $D(x)$.

1. If $n < m$, then the graph of f has the line $y = 0$ (the x-axis) as a horizontal asymptote.
2. If $n = m$ then the graph of f has the line $y = \frac{a_n}{b_m}$ as a horzontal asymptote.
3. if $n > m$ then the graph of f has no horizontal asymptote.

Example 4.1.2. Find the domain of the function and identify any horizontal and vertical asymptotes. Sketch a graph for each.

1. $f(x) = \dfrac{x-4}{(x-2)^2}$

2. $f(x) = \dfrac{x-4}{1+2x}$

3. $f(x) = \dfrac{(x-4)^2}{(x+1)^2}$

4. $f(x) = \dfrac{(x-4)^2}{x-3}$

5. $f(x) = \dfrac{1}{x} + 2$

4.2 Graphing Rational Functions

Example 4.2.1. Let

$$f(x) = \frac{3x^2 + 4x + 1}{3x^2 + 11x - 20}$$

sketch the graph with the

1. y-intercept(s)
2. x-intercept(s)
3. vertical asymptote(s)
4. horizontal asymptote(s)

Example 4.2.2. Sketch and write an equation for a rational function with:

1. Vertical asymptotes at $x = 5$ and $x = -5$
2. x-intercepts at $x = 1$ and $x = 2$
3. y intercept at 3

4.3 Rational Equations and Applications

Example 4.3.1. Suppose f varies inversely with g and that $f = 36$ when $g = 6$. What is the value of f when $g = 12$?

Example 4.3.2. Solve the equation $$\frac{2}{x} = \frac{4}{3x} - 5.$$

Example 4.3.3. Solve the equation $$\frac{8}{x+1} - \frac{5}{2} = \frac{4}{3x+3}.$$

Example 4.3.4. Solve the equation $$\frac{x}{2x-4} - 9 = \frac{1}{x-2}.$$

Example 4.3.5. Solve the equation $$\frac{x+1}{x-1} = \frac{-1}{x+3} + \frac{8}{x^2+2x-3}.$$

4.3.1 Rational Inequalities

Example 4.3.6. Solve the inequality $$\frac{x+4}{x+7} < -3.$$

Example 4.3.7. Solve the inequality $$\frac{x-2}{x^2-25} < -3.$$