3.1 Graphs of Polynomials

Parts of a Polynomial

Definition 3.1. A **polynomial of degree** n is a function of the form:

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$$

where $a_n \neq 0$ and the **coefficients** $a_n, a_{n-1}, \ldots, a_1, a_0$ are real numbers.

The **leading term** is $a_n x^n$.

The **leading coefficient** is a_n.

The **degree** of the polynomial is n.

The **constant term** is a_0

Example 3.1.1. Find the leading term, the leading coefficient, the degree and the constant term for the polynomial

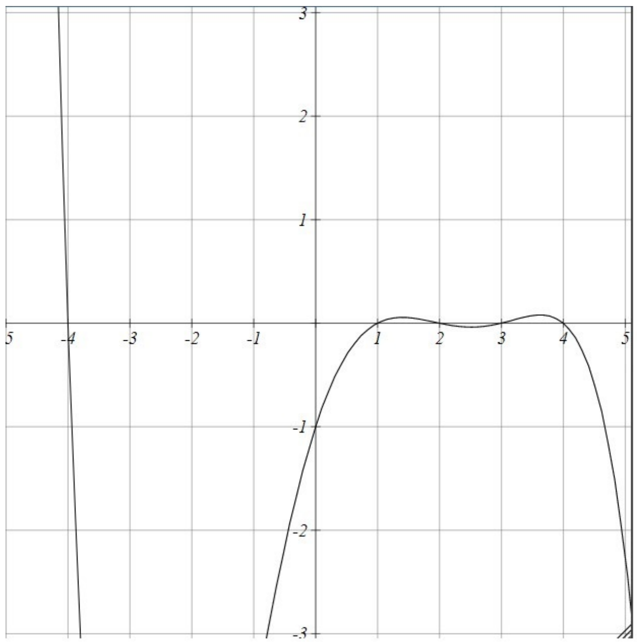
$$f(x) = x^4 - x^3 - 20x^2 - 3x^6 + 12$$

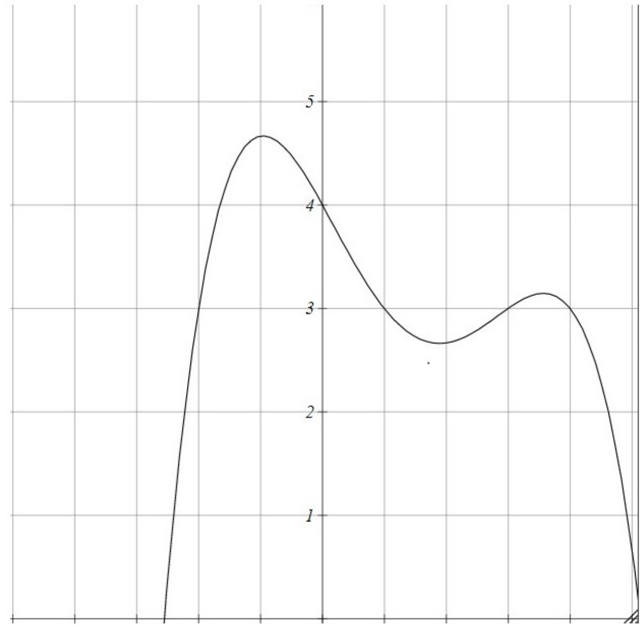
3.1.1 General shape and long run behavior.

Odd exponent leading term: x, x^3, x^5, etc

Even exponent leading term: x^2, x^4, x^6, etc

Example 3.1.2. Find the minimum degree for the following





Example 3.1.3. Find all zeros of $f(x) = 5x^4 - x^3 - 20x^2$

To find zeros you must set the function equal to zero. Do not do this unless you are looking for zeros.

$$x^4 - x^3 - 20x^2 = 0$$
$$x^2(x^2 - x - 20) = 0$$
$$x^2(x - 5)(x + 4) = 0$$

Now we have three things multiplied together that equal zero so one of them must be zero.

$$x^2 = 0 \quad \text{OR} \quad x - 5 = 0 \quad \text{OR} \quad x + 4 = 0.$$

Example 3.1.4. Find the zeros and intercepts of

$$f(r) = 49 - r^3$$

Example 3.1.5. Find the zeros and intercepts of

$$h(x) = 2x^4 - 2x^2 - 40$$

Example 3.1.6. Find a polynomial of degree 3 with the following zeros: $x =$ -2, 0, 6.

3.1.2 Multiplicity

Multiplicity

Definition 3.2. Suppose $f(x)$ is a polynomial function and m is a natural number. If $(x-c)^m$ is a factor of $f(x)$ but $(x-c)^{m+1}$ is not, then we say $x=c$ is a zero of **multiplicity** m.

The Role of Multiplicity in Graphing

Suppose $f(x)$ is a polynomial function and $x = c$ is a zero of multiplicity m.

If m is even, the graph of $y = f(x)$ touches and rebounds from the x-axis at $(c, 0)$.

If m is odd, the graph of $y = f(x)$ crosses through the x-axis at $(c, 0)$.

Example 3.1.7. Create a polynomial $P(x)$ which has the desired characteristics. Leave the polynomial in factored form. Sketch.

- degree 4
- root of multiplicity 2 at $x = 1$
- roots of multiplicity 1 at $x = 0$ and $x = -3$.
- the graph passes through the point (2,20)

Example 3.1.8. Find a formula for the polynomial $P(x)$ which has the desired characteristics. Leave the polynomial in factored form. Sketch.

- degree 9
- leading coefficient 1
- root of multiplicity 5 at $x = 0$
- root of multiplicity 2 at $x = 11$
- root of multiplicity 2 at $x = -5$

3.2 The Factor Theorem and the Remainder Theorem

Example 3.2.1. Simplify this fraction with long division:

$$\frac{x^4 + 5x^3 + 6x^2 - x - 2}{x + 2}$$

So we know that

$$\frac{x^4 + 5x^3 + 6x^2 - x - 2}{x + 2} = x^3 + 3x^2 - 1.$$

More specifically we have a factor of $f(x) = x^4 + 5x^3 + 6x^2 - x - 2$.

$$\boxed{x^4 + 5x^3 + 6x^2 - x - 2 = (x^3 + 3x^2 - 1)(x + 2).}$$

Example 3.2.2. $\dfrac{5x^3 + 18x^2 + 8x - 6}{x + 3}$

So we know that

$$\frac{5x^3 + 18x^2 + 8x - 6}{x + 3} = 5x^2 + 3x - 1 + \frac{-3}{x + 3}.$$

If we write it with no denominators we get an expression of the form.

$$\boxed{5x^3 + 18x^2 + 8x - 6 = (5x^2 + 3x - 1)(x + 3) - 3}$$

This form illustrates the theorem known as **The Division Algorithm** which states that any two polynomials $P(x)$ and $D(x)$ (where $D(x)$ is of lower degree) can be written as

$$P(x) = D(x) \cdot q(x) + r(x)$$

The Division Algorithm

If $P(x)$ and $D(x)$ are polynomials such that $D(x) \neq 0$ and the degree of $D(x)$ is less than or equal to the degree of $P(x)$, there exist unique polynomials $q(x)$ and $r(x)$ such that

$$P(x) = D(x) \cdot q(x) + r(x)$$

OR

$$\frac{P(x)}{D(x)} = q(x) + \frac{r(x)}{D(x)}$$

Synthetic Division (The easy way to divide)

Example 3.2.3. Divide using synthetic division:

$$\frac{5x^3 + 18x^2 + 8x - 6}{x + 3}$$

The Remainder Theorem

Suppose $p(x)$ is a polynomial of degree at least 1 and c is a real number. When $p(x)$ is divided by $x - c$ the remainder is $p(c)$.

Example 3.2.4. Use the remainder theorem to find the value of $p(-3)$ where

$$p(x) = 5x^3 + 18x^2 + 8x - 6$$

3.3 Real Zeros of a Polynomial

Example 3.3.1. Suppose we know that $x = -2$ is a zero of the polynomial

$$f(x) = 2x^3 + x^2 - 5x + 2$$

. Find all the zeros of the polynomial and write it in factored form.

Solution: Since we know that $x = -2$ is a zero then we know that

$$f(x) = 2x^3 + x^2 - 5x + 2 = (x + 2) \cdot q(x)$$

and we can use synthetic division to find $q(x)$.

Example 3.3.2. Use synthetic division to divide $\dfrac{-3x^4}{x+2}$

Example 3.3.3. Given that $(x + 2)$ and $(x - 4)$ are factors of

$$f(x) = 8x^4 - 14x^3 - 71x^2 - 10x + 24$$

find all the zeros of $f(x)$.

The Rational Roots Test

If the polynomial $f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_2 x^2 + a_1 x + a_0$ has *integer* coefficients, every rational zero of f has the form

$$\text{Rational zero} = \frac{p}{q}$$

where p and q have no common factors other than 1, and

$p =$ a factor of the constant term a_0

$q =$ a factor of the constant term a_n.

Example 3.3.4. Use the rational roots test to find all the real zeros

$g(x) = 6x^3 - 7x^2 + 1$ (ans. 1, 1/2, -1/3)

Example 3.3.5. Use the rational roots test to find all the real zeros
$p(x) = 12x^3 - 35x^2 + 7x + 30$ (ans. 2, 5/3, -3/4)

3.4 Complex Zeros and the Fundamental Theorem of Algebra

Q: What is a complex number?

A: It is a number of the form $a + bi$ where a and b are real numbers and $i^2 = -1$.

- a is called the **real part**
- b is called the **imaginary part**.
- The **conjugate** of $a + bi$ is $a - bi$.

Properties

1. $a + b\ i = c + d\ i \quad \Leftrightarrow \quad a = c$ and $b = d$
2. $(a + b\ i) + (c + d\ i) = (a + c) + (b + d)\ i$
3. $(a+b\ i)\cdot(c+d\ i) = ac+ad\ i+bc\ i+bd\ i^2 = (ac-bd)+(ad+bc)\ i$
4. $(a + b\ i)(a - b\ i) = a^2 + b^2$

Example 3.4.1.

a. $(4 + i) + (5 + 3i)$

b. $(2i + 7) - 2i$

c. $(3 + 2i) + (4 - i) - (7 + i)$

d. $(5 + 2i)(4 - 3i)$

e. $\dfrac{i}{3+i}$

f. $\dfrac{3-5i}{2-i}$

g. $(3-\sqrt{-4})+(-8+\sqrt{-25})$

h. $(\sqrt{-5})(\sqrt{-5})$

i. $(2-\sqrt{-1})(5+\sqrt{-9})$

j. $\dfrac{1}{3i}$

Complex numbers in quadratic equations

Example 3.4.2. Solve for x: $x^2 + 6x + 10 = 0$

Example 3.4.3. Solve for x: $x^2 + 1 = 0$

Example 3.4.4. Solve for x: $9x^2 - 6x + 37 = 0$

The Linear Factorization Theorem

If $f(x)$ is a polynomials of degree n, where $n > 0$, then f has precisely n linear factors

$$f(x) = a_n(x - c_1)(x - c_2) \cdots (x - c_n)$$

where $c_1, c_2, \ldots, c_n$ are complex numbers.

Example 3.4.5. Find all the zeros of $f(x) = (x + 5)(x - 8)^2$

Example 3.4.6. Find all the zeros of

$$f(t) = (t - 3)(t - 2)(t - 3i)(t + 3i)$$

Example 3.4.7. Find a polynomial function of degree 7 with integer coefficients that has ONLY the zeros $c = 4$, $c = -3i$ and $c = 3i$

Example 3.4.8. Find all zeros of $f(x) = 4x^3 + 12x^2 + 11x + 6$ given that $x = -2$ is one of the zeros.

Example 3.4.9. Find all the roots of $f(x) = x^3 - 7x^2 - x + 87$ knowing that $5 + 2i$ is a zero.