2.1 Linear Equations in Two Variables

The simplest mathematical model is the **linear equation in two variables**. The standard form is (**slope-intercept**)

$$\boxed{y = mx + b}$$

where m is the slope and b is the y-intercept. You will recall that

$$\text{slope} = \frac{\text{rise}}{\text{run}} = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1}$$

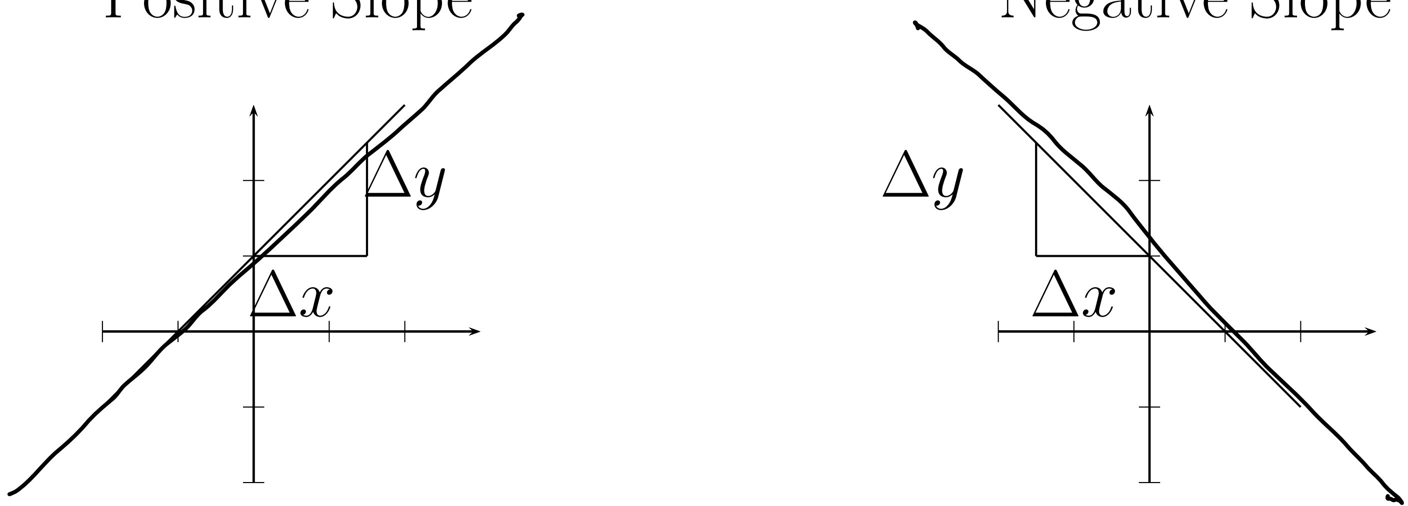
The slope is the amount of vertical change relative to the horizontal change. Sometimes we think of it as the "change in y" over "change in x".

To calculate the slope between two points (x_1, y_1) and (x_2, y_2) the formula is:

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{\Delta y}{\Delta x} = \frac{\text{rise}}{\text{run}}.$$

Positive Slope

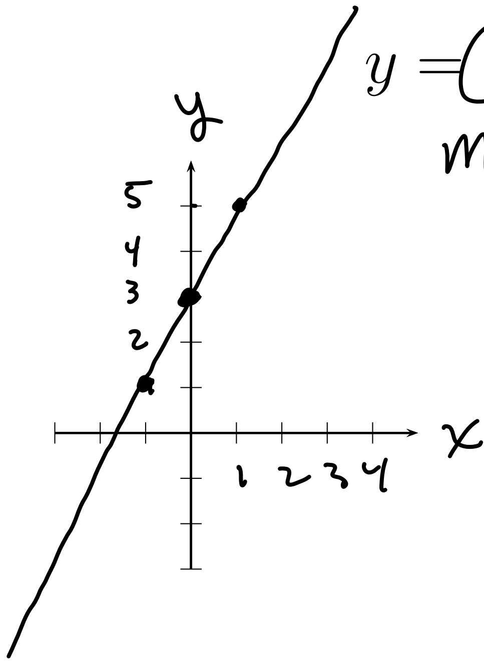
Negative Slope



Example 2.1.1. Sketch the graphs of the following two functions:

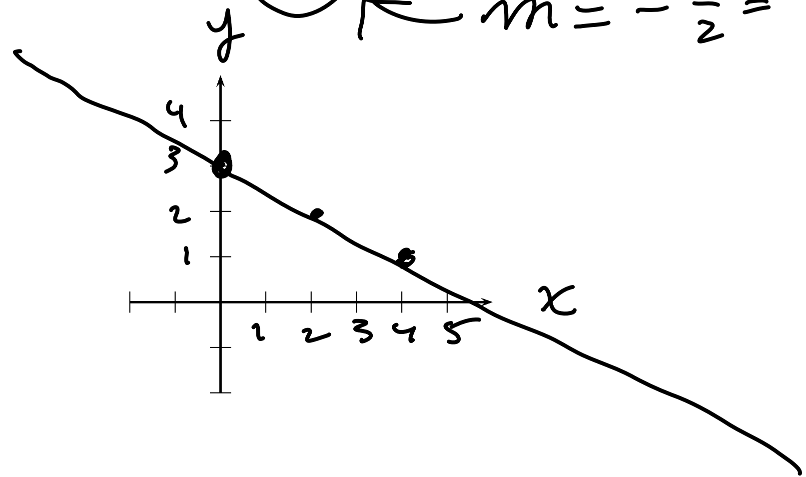
$$y = 2x + 3 \quad \text{y-int}$$

$$m = \frac{2}{1} = \frac{\text{rise}}{\text{run}}$$



$$y = -\frac{1}{2}x + 3.$$

$$m = -\frac{1}{2} = \frac{\text{rise}}{\text{run}}$$



Example 2.1.2. Find the slope between the following pairs of points.

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

(x_1, y_1) (x_2, y_2)

(a). (-3, 0) and (4, 4)

(b). (-3, 1) and (4, 1)

(c). (-3, 1) and (-3,4)

a) $m = \frac{4-0}{4-(-3)} = \frac{4}{7}$

b) $m = \frac{1-1}{4-(-3)} = \frac{0}{7} = \boxed{0}$ zero slope

c) $m = \frac{4-1}{-3-(-3)} = \frac{3}{0}$ $\boxed{\text{DNE}}$

undefined slope

2.1.1 Point-Slope Form

slope

$$\boxed{y - y_1 = m(x - x_1)}$$

any point (x_1, y_1)

You always need **two** things:

1. a point: (x_1, y_1) AND
2. a slope m.

(x_1, y_1)

(x_2, y_2) **Example 2.1.3.** Write the equation of the line through (-3,0) and (4, -4). Write the equation in the point slope form and the slope-intecept form. $y = mx + b$

$$\text{slope} = m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{0 - (-4)}{-3 - 4} = \frac{-4 - 0}{4 - (-3)} = -\frac{4}{7}$$

$y - y_1 = m(x - x_1)$

$$\boxed{y - 0 = -\frac{4}{7}(x - (-3))}$$

$$\boxed{y = -\frac{4}{7}x - \frac{12}{7}}$$

Horizontal line equation

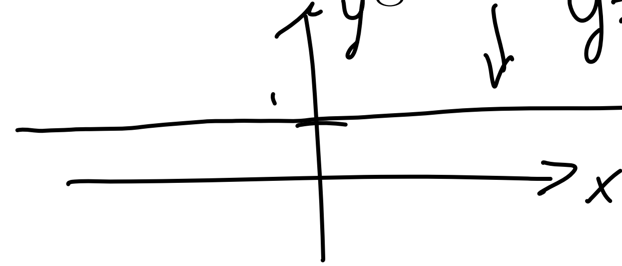
$\downarrow$ $y =$ some number

Example 2.1.4. Write the equation of the lines through

(x_1, y_1)

(a). (-3, 1) and (4, 1) $\quad m = 0$

(b). (-3, 1) and (-3,4) $\quad m =$ DNE

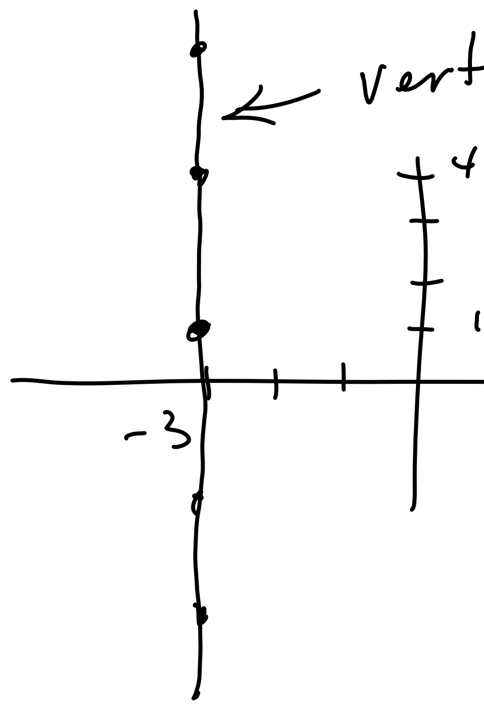


a) $y - y_1 = m(x - x_1)$ $\qquad x_1 = -3, \; y_1 = 1$

$y - 1 = 0(x - (-3))$

$\boxed{y = 1}$

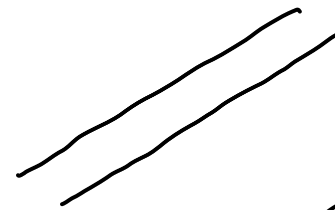
b) $\boxed{x = -3}$



vertical line equation

$x =$ some number

2.1.2 Parallel and Perpendicular Lines

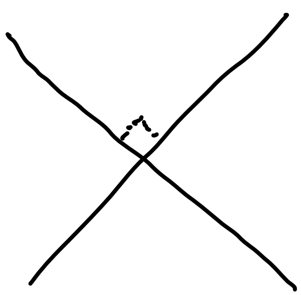


Parallel lines have the same slope.

If $y = m_1x + b_1$ is parallel to $y = m_2x + b_2$ then $m_1 = m_2$.

Perpendicular lines have negative reciprocal slopes.

If $y = m_1x + b_1$ is perpendicular to $y = m_2x + b_2$ then $m_1 = -\dfrac{1}{m_2}$.



Ex: slope $m_1 = \dfrac{2}{1}$

slope of perpendicular line

slope $m_2 = -\dfrac{1}{2}$

Example 2.1.5. Write the equations of the lines parallel and perpendicular to $-4x + 2y = 3$ passing through the point (2, 1).

Need 2 things for a line

1. point $(2, 1) = (x_1, y_1)$

2. slope

a) parallel line $m_1 = 2$

$y - y_1 = m(x - x_1)$

$y - 1 = 2(x - 2)$

$y = 2x - 4 + 1$

$\boxed{y = 2x - 3}$

$$-4x + 2y = 3$$
$$+4x \qquad\qquad +4x$$
$$\frac{2y}{2} = \frac{3}{2} + \frac{4x}{2}$$
$$y = 2x + \frac{3}{2}$$
$$m = 2$$

b) Perpendicular line $\perp$

$m_\perp = -\frac{1}{2}$

$y - y_1 = m(x - x_1)$

$1 + y - 1 = -\frac{1}{2}(x - 2) + 1$

$\boxed{y = -\frac{1}{2}x + 2}$

2.2 Absolute Value Functions

Definition 2.1. The absolute value of a real number x, denoted $|x|$

$$|x| = \boxed{\begin{cases} -x, & \text{if } x < 0 \\ x, & \text{if } x \geq 0 \end{cases}}$$

Absolute Value Properties

$|4| = 4$

$|-4| = 4$

1. **Product rule:** $|ab| = |a||b|$
2. **Power rule:** $|a^n| = |a|^n$
3. **Quotient rule:** $\left|\frac{a}{b}\right| = \frac{|a|}{|b|}$
4. **Equality property 1:** $|x| = 0$ if and only if $x = 0$
5. **Equality property 2:** For $c > 0$, $|x| = c$ if and only if $x = c$ or $x = -c$.
6. **Equality property 1:** For $c < 0$, $|x| = c$ has no solution.

$|x| = -4$

$|1-5| = |-4| = 4$ ✓

An equation with an absolute value is always TWO equations:

$$| x - 5 | = 4 \quad \Longrightarrow \quad x - 5 = 4 \text{ OR } -(x-5) = 4$$

$x - 5 = 4$ or $x - 5 = -4$

$x = 9$ $\quad$ $x = 1$

$|9-5| = |4| = 4$ ✓

Example 2.2.1. $-4 + 2 \mid 4x - 4 \mid = 10$

We start by getting the absolute value by itself on one side of the equation.

$-4 + 2|4x-4| = 10$

$+4 \qquad\qquad +4$

$\frac{2|4x-4|}{2} = \frac{14}{2}$

$|4x-4| = 7$

$4x - 4 = 7$ OR $4x - 4 = -7$

$+4 \quad +4$

$4x = 11 \qquad 4x = -3$

$\boxed{x = \frac{11}{4} \text{ OR } x = -\frac{3}{4}}$

$x = -\frac{3}{4}$

$-4 + 2|4(-\frac{3}{4}) - 4| = -4 + 2|-7| = 10$ ✓

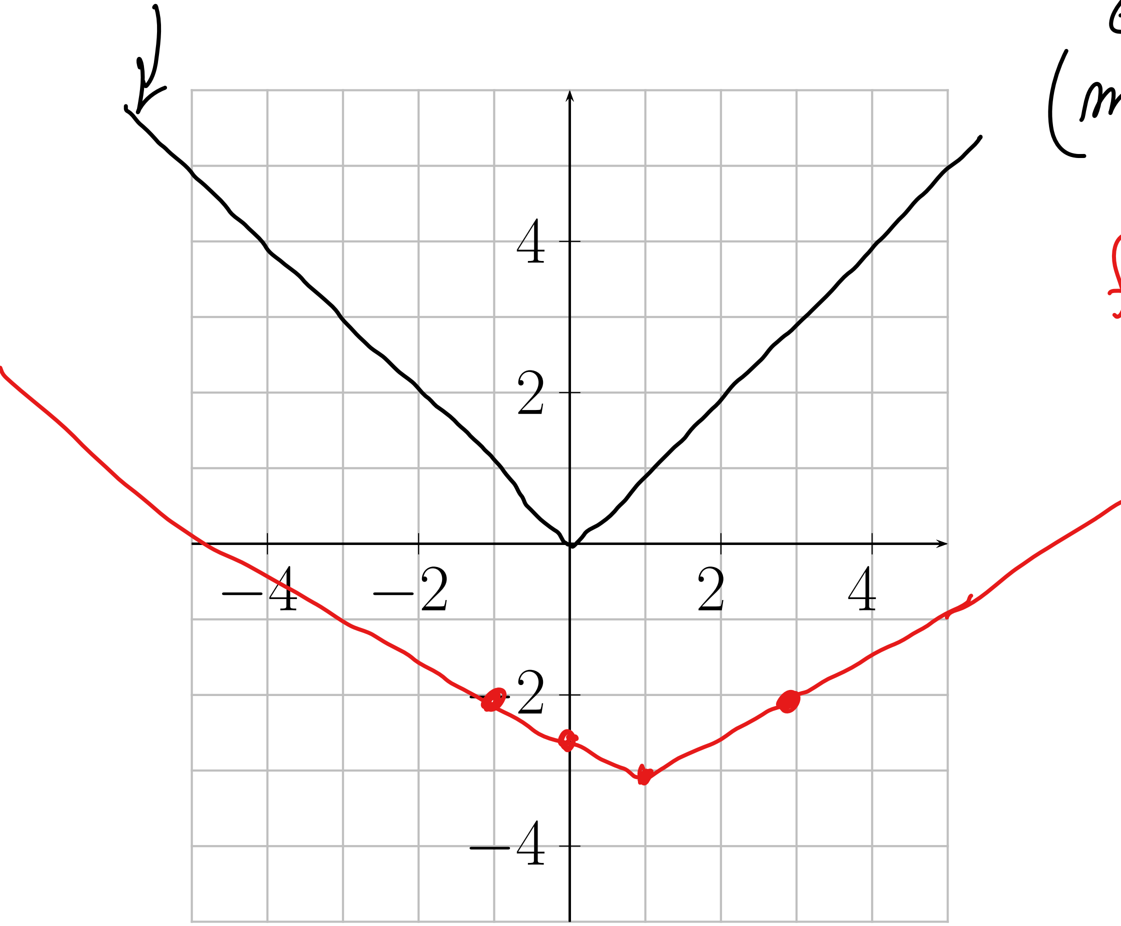
Example 2.2.2. Sketch a graph of $f(x) = \frac{1}{2}|x - 1| - 3$

expand (make wider)

move right

move down

$y = |x|$



$$f(3) = \frac{1}{2}|3 - 1| - 3$$

$$= \frac{1}{2}(2) - 3$$

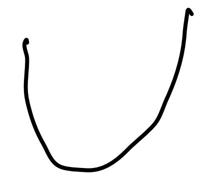
$$= 1 - 3$$

$$= -2$$

2.3 Quadratic Functions

Definition 2.2. Let a, b, and c be real numbers with $a \neq 0$. The function

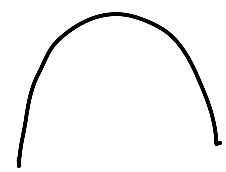
$$f(x) = ax^2 + bx + c$$



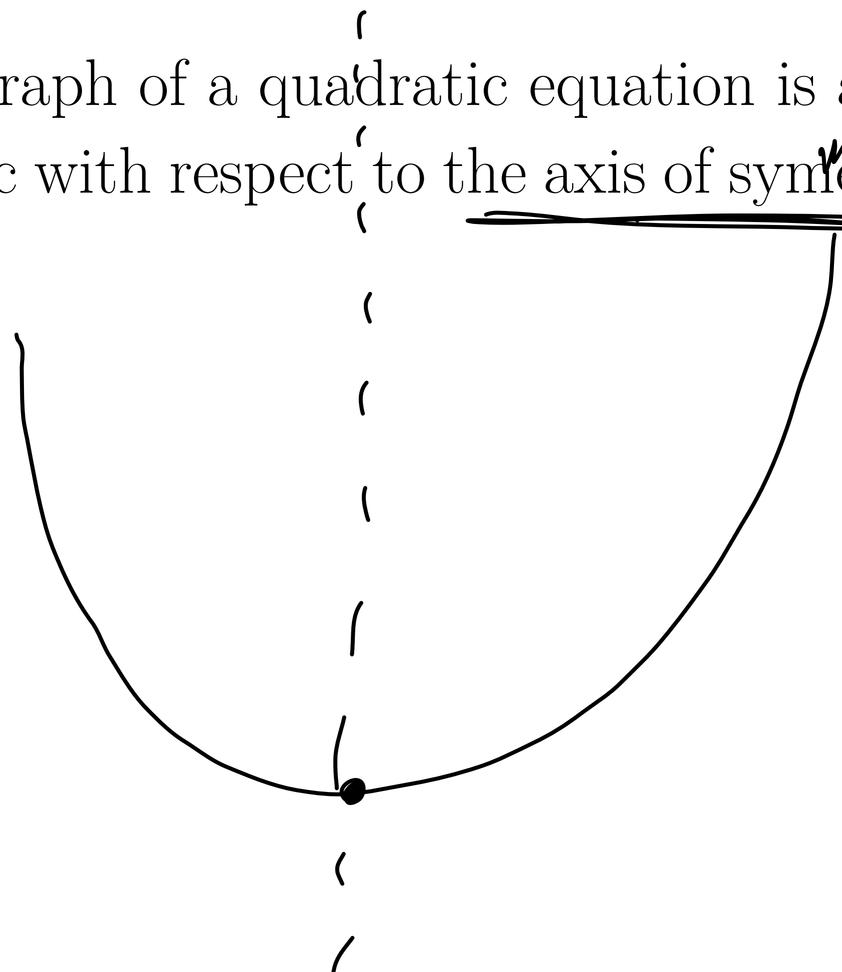
x^2

is called a **quadratic equation**.

The graph of a quadratic equation is a parabola. All parabolas are symetric with respect to the axis of symetry which passes through the vertex.



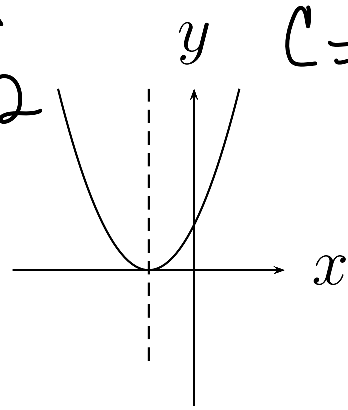
$-x^2$

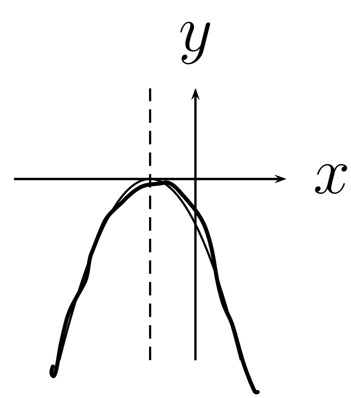


Example 2.3.1.

$f(x) = x^2 + 2x + 1$ OR $f(x) = -x^2 - 2x - 1$

$a=1$ $b=2$ $c=1$ $a=-1$





$a > 0$ graph opens up

$a < 0$ graph opens down

The Standard Form of a Parabola

moves up/down

y-Coord

$$f(x) = a(x - h)^2 + k$$

Vertex is located at (h, k)

moves right/left x coord.

- if $a > 0$ graph opens up
- if $a < 0$ graph opens down

Alternate form

If

$$y = f(x) = ax^2 + bx + c$$

then the

$$\text{vertex} = \left(\frac{-b}{2a}, f\left(\frac{-b}{2a} \right) \right).$$

x-coord y-coord

To Graph a Parabola

1. Find the vertex
2. Find the x - intercepts
3. Determine if it opens up or down
4. Sketch

#2 x-intercepts: set $y = 0 = ax^2 + bx + c$

1. quadratic formula

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

2. factoring $(x-A)(x-B) = 0$

3. completing the square.

(Later)

Example 2.3.2. Graph $f(x) = 2x^2 - 12x - 14$.

$x = \frac{-b}{2a}$ $a = 2$ $b = -12$ $c = -14$

Step 1: Vertex:

$$x = \frac{-(-12)}{2(2)} = 3$$ ✓

$$y = f(3) = 2(9) - 12(3) - 14 = -32$$

So the coordinates of the vertex are $(x, y) = (3, -32)$

Step 2: x - intecepts:

$$0 = 2x^2 - 12x - 14$$

$$0 = 2(x^2 - 6x - 7)$$

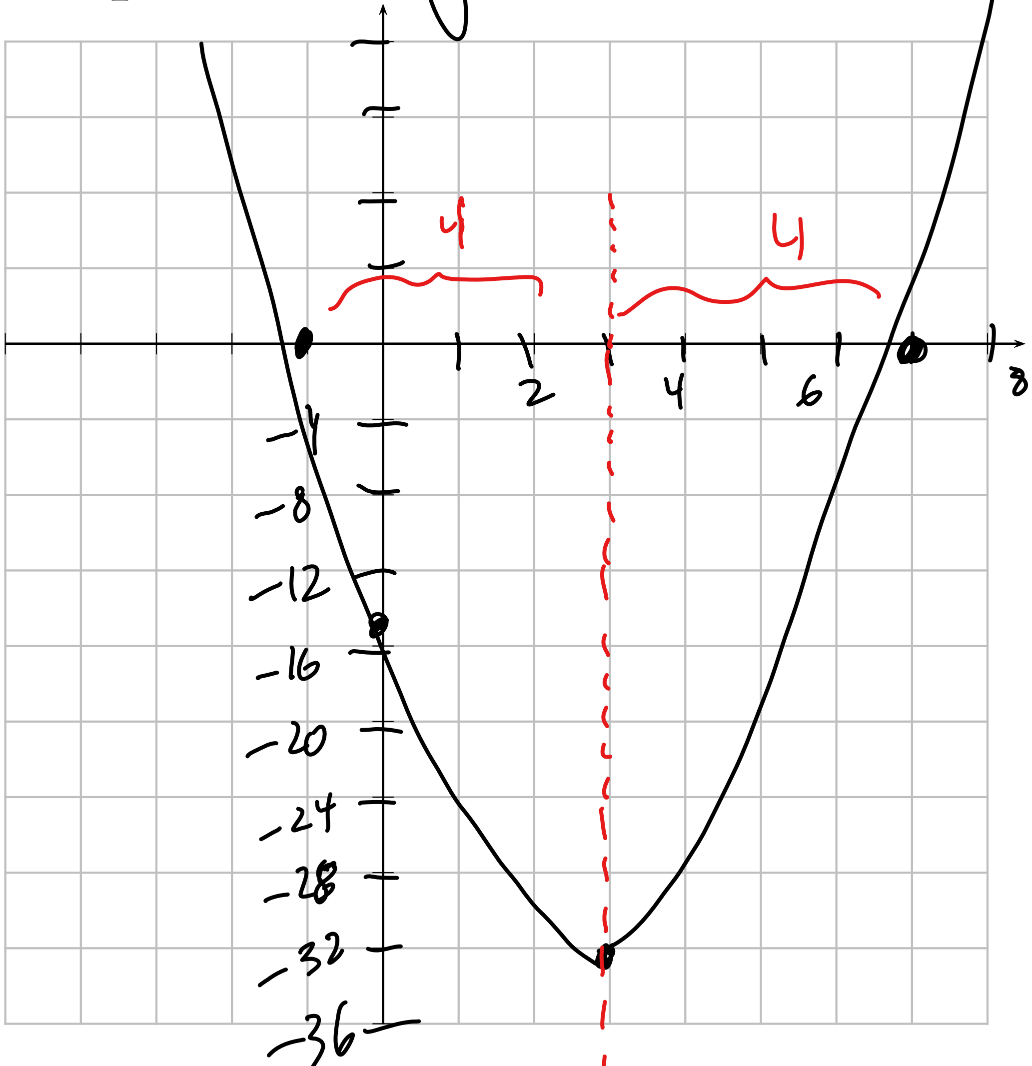
$$0 = 2(x-7)(x+1)$$

$x - 7 = 0$ OR $x + 1 = 0$

$x = 7$ $x = -1$

Step 4: Sketch

$y = 2x^2 - 12x - 14$



Vertex

$(3, -32)$

x-int

$x = -1, 7$

Axis of symmetry

$x = 3$

Example 2.3.3. Graph $f(x) = (x-6)^2 - 3$

opposit sign

Vertex $(6, -3)$

same sign

x-intercepts $(x-6)^2 - 3 = 0$

$$\underline{\qquad +3 \qquad +3}$$

$$\sqrt{(x-6)^2} = \pm\sqrt{3}$$

$$x - 6 = \pm\sqrt{3}$$

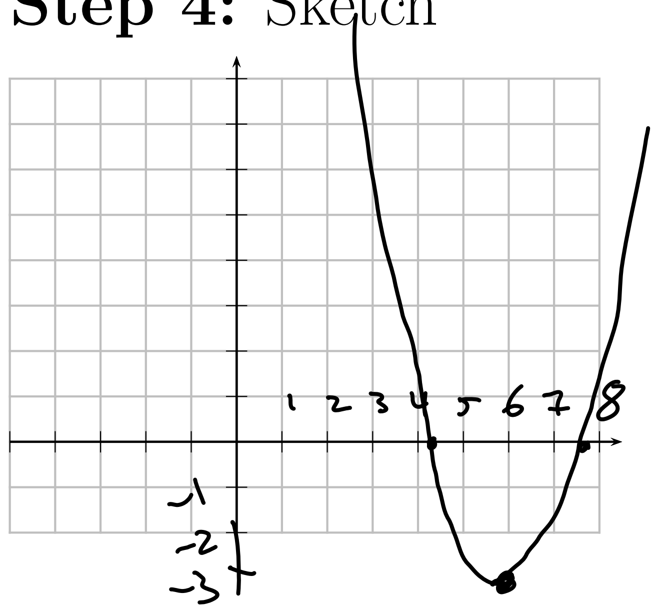
$$x = 6 \pm \sqrt{3}$$

$$x = 6+\sqrt{3}, \quad 6-\sqrt{3}$$

$$\approx 7.7 \qquad \approx 4.3$$

Not the answers

Step 4: Sketch



Example 2.3.4. Find the standard form for a parabola that has (0,1) as its vertex and passes through the point (1,0)

$$y = f(x) = a(x-h)^2 + k$$

Vertex $(h, k) = (0, 1)$

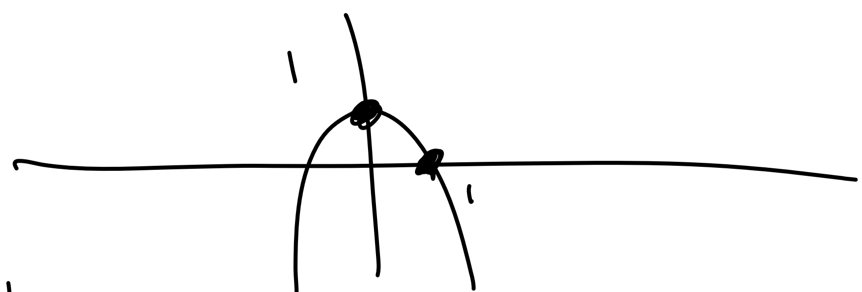
$h = 0 \quad k = 1$

$(1, 0)$

$x = 1 \quad f(1) = 0$

$$f(x) = a(x-0)^2 + 1$$

$$f(x) = ax^2 + 1$$



$$f(1) = 0 = a(1) + 1$$

$a = -1$

$$\boxed{f(x) = -x^2 + 1}$$

Example 2.3.5. Convert to standard form $f(x) = x^2 + 6x + 5$ and graph.

$$f(x) = a\boxed{(x-h)^2} + k$$

completing the square: want $\underbrace{(x+a)^2}$

have $x^2 \boxed{+b}x$ $\xrightarrow{\text{convert}}$ $x^2 \boxed{+2a}x + a^2$

Need this

take half of b & square it:

$$\left(\frac{b}{2}\right)^2$$

$$x^2 + bx + \left(\frac{b}{2}\right)^2 - \left(\frac{b}{2}\right)^2 = \left(x + \frac{b}{2}\right)^2 - \left(\frac{b}{2}\right)^2$$

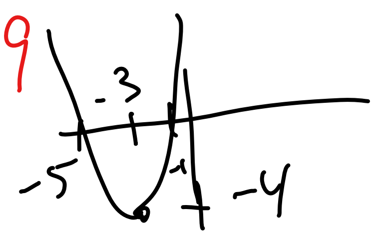
$$x^2 + 6x + 5 = x^2 + 6x + 9 - 9 + 5 = \boxed{(x+3)^2 - 4}$$

$(x-h)^2 + k$

$h = -3$

$k = -4$

$$\left(\frac{6}{2}\right)^2 = 3^2 = 9$$



Standard form

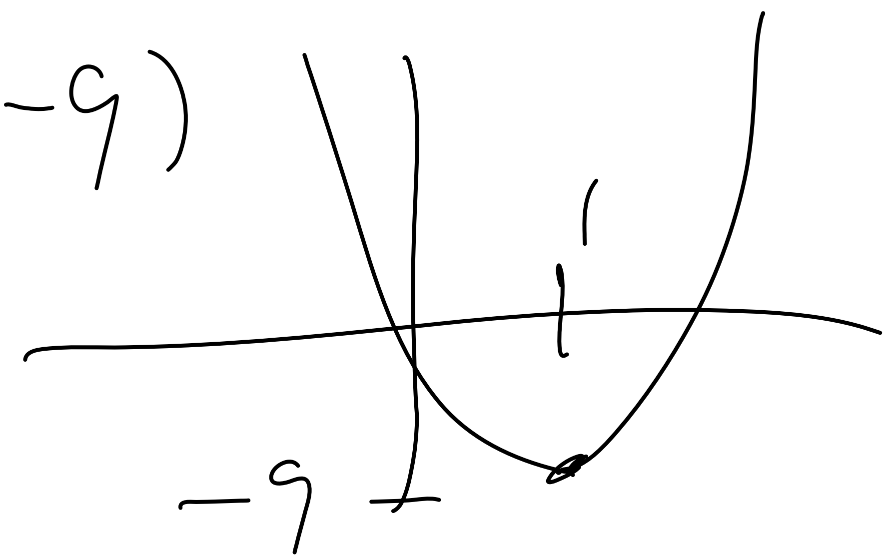
Example 2.3.6. Convert to standard form $f(x) = x^2 - 2x - 8$ and graph.

$$f(x) = x^2 - 2x + 1 - 1 - 8$$

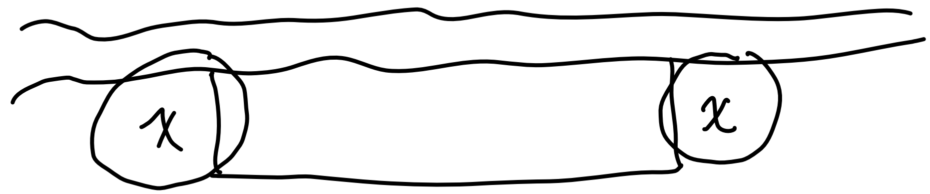
leave a space

$$f(x) = (x-1)^2 - 9 = (x-h)^2 + k$$

Vertex $(1, -9)$



Example 2.3.7. A rancher has 260 yards of fence with which to enclose three sides of a rectangular meadow (the fourth side is a river and will not require fencing). Find the dimensions of the meadow with the largest possible area.



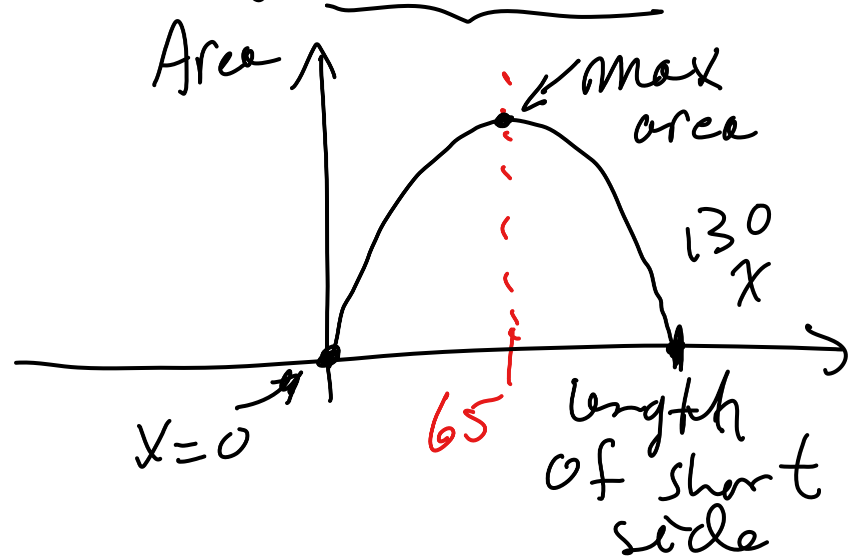
$y = 260 - 2x$

$A = \text{Length} \cdot \text{width}$

$A = (260 - 2x)\,x = 260x - 2x^2$

Vertex $\quad x = \dfrac{-b}{2a} = \dfrac{-260}{2(-2)}$

$x = 65$

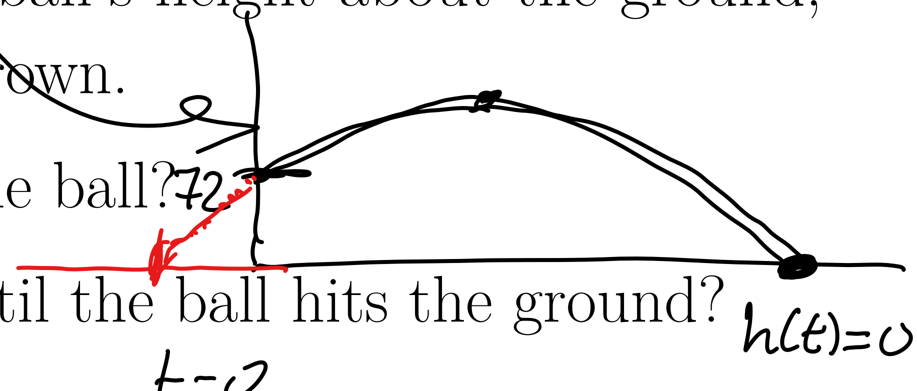


$y = 260 - 2(65)$

$y = 130$

Max Area $= 65(130) = \boxed{8450 \text{ sq yds}}$

Example 2.3.8. A person standing close to the edge on top of a 72-foot building throws a ball vertically upward. The quadratic function $h(t) = -16t^2 + 84t + 72$ models the ball's height about the ground, $h(t)$, in feet, t seconds after it was thrown.



1. What is the maximum height of the ball?
2. How many seconds does it take until the ball hits the ground?

vertex of parabola

$t = -\frac{b}{2a} = \frac{-84}{-(-16)2} = \#$ $h(\#) = \text{height}$

set $h(t) = 0 = -16t^2 + 84t + 72$

$0 = -4(4t^2 - 21t - 18)$

$t = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{+21 \pm \sqrt{21^2 - 4(4)(-18)}}{2(4)}$

2.4 Solving Inequalities with Absolute Value and Quadratic Functions

2.4.1 Graphing inequalities

$2x+3 < 5$

Properties of Inequalities

1. Transitive: $\underline{a < b}$ and $\underline{b < c} \implies \underline{a < c}$
2. Addition of Constants: If $a < b$ then $a + c < b + c$
3. Addition: If $a < b$ and $c < d$ then $a + c < b + d$
4. Multiplication by a constant:

   $1 < 2$

   If $c > 0$ and $a < b$ then $ac < bc$

   If $c < 0$ and $a < b$ then $ac > bc$

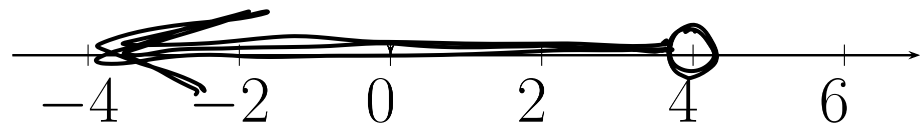
   $(-1)(1) \quad 2(-1)$

   $-1 > -2$

NOTE: If you multiply or divide by a negative number you reverse the order of the inequality.

2.4.2 Solving linear inequalities

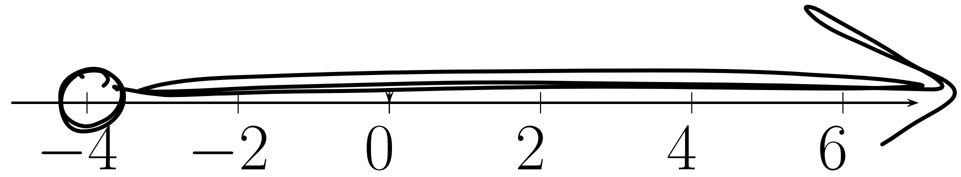
Example 2.4.1. $10x < 40$



$$\frac{10x}{10} < \frac{40}{10}$$

$$x < 4$$

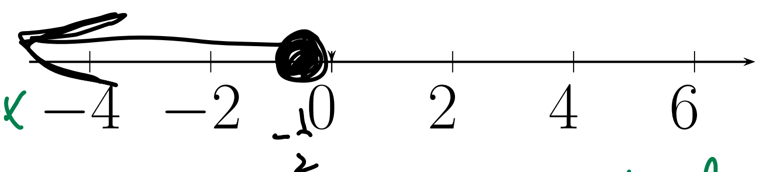
Example 2.4.2. $-10x < 40$



$$\frac{-10x}{-10} < \frac{40}{-10}$$

$$x > -4$$

Example 2.4.3. $4(x+1) \leq 2x+3$

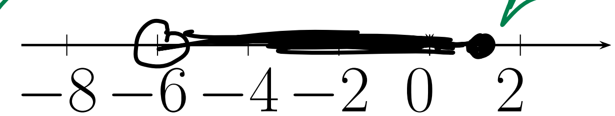


$$-2x-4+4x+4 \le 2x+3-4-2x$$

$$\frac{2x}{2} \le \frac{-1}{2} \qquad x \le -\frac{1}{2}$$

Example 2.4.4. $-8 \leq -(3x+5) < 13$

Included



$$-8 \le -3x-5 < 13$$

$$+5 \qquad +5 \qquad +5$$

$$\frac{-3}{-3} \le \frac{-3x}{-3} < \frac{18}{-3}$$

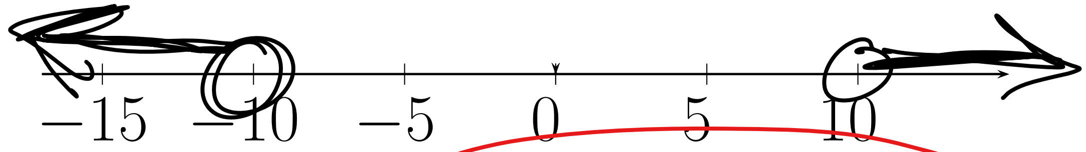
not included

$$1 \ge x > -6$$

2.4.3 Absolute value and inequalities

Absolute value is still two equations

Example 2.4.5. $\left|\frac{x}{2}\right| > 5$



$\frac{x}{2} > 5$ OR $\left(-\frac{x}{2} > 5\right)(-2)$

$x > 10$ $\quad$ $x < -10$

$-100 = x$ $\quad$ $\left|\frac{-100}{2}\right| = 50 > 5$ ✓

Example 2.4.6. $|x - 7| < 5$

center $\quad$ dist.

$-5 < x - 7 < 5$

$+7 \quad +7 \quad +7$

$2 < x < 12$

all numbers within 5 units of 7

$7-5=2$ $\quad$ 7 $\quad$ $7+5=12$

$x = 4$ $\quad$ $|4 - 7| = 3 < 5$ ✓

checking to make myself feel good.

Question: What does $|x - 2| < 5$ mean?

Answer: All real numbers within five units of two.

center distance

So all real numbers within 5 units of 8 would be written as:

$$|x - 8| < 5$$

And all real numbers at least 5 units from 8 would be written as:

$$|x - 8| \geq 5$$

$\rightarrow x^2-5$ negative

2.4.4 Solving polynomial inequalities

$-\sqrt{5} < x < \sqrt{5}$

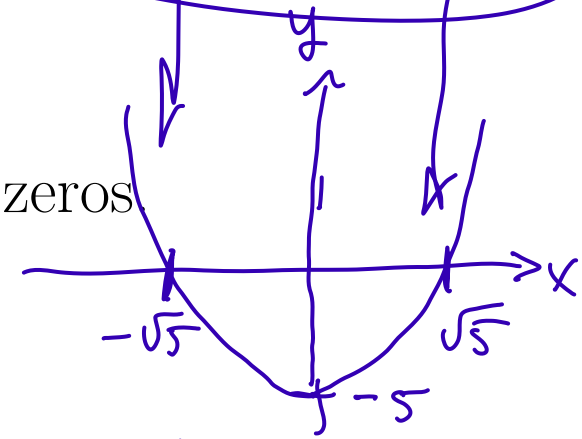
Example 2.4.7. $x^2 < 5$

$x^2-5<0$

Step 1: Set equation equal to zero and find the zeros.

$y = x^2-5 = 0$

$x=\sqrt{5}$, $x=-\sqrt{5}$



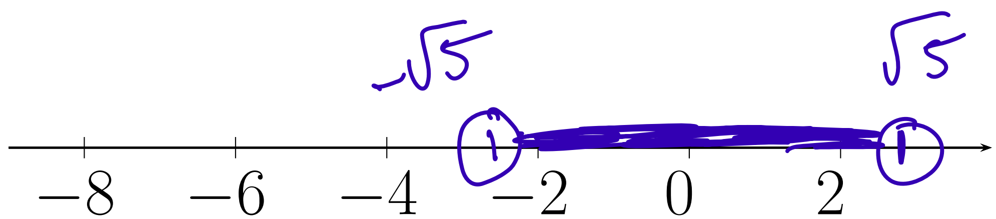
negative numbers

Step 2: Set up a table of signs

	$-\sqrt{5}$		$\sqrt{5}$	
x-value	$x=-10$	0		10
x^2-5	$+$	$-$		$+$

↑ negative

Step 3: Find where the table gives negative values and write the solution

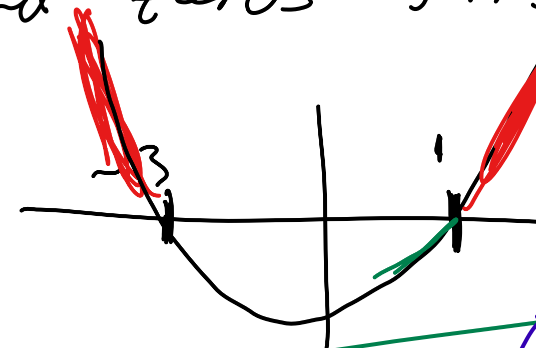


Example 2.4.8. $x^2 + 2x - 3 \geq 0$

$x^2 + 2x - 3 = 0 \leftarrow$ Find zeros first.

$(x+3)(x-1) = 0$

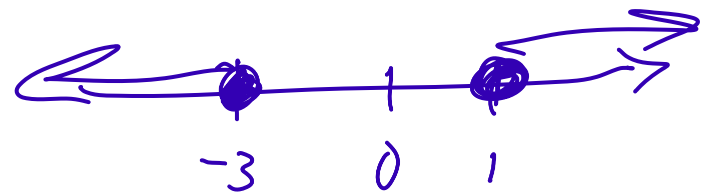
$x = -3, \quad x = 1$



$-3 \qquad\qquad 1$

x	-5	0	2
$(x+3)(x-1)$	$(-)(-)$	$(+)(-)$	$(+)(+)$
	$+$	$-$	$+$

$\boxed{x \leq -3 \qquad x \geq 1}$

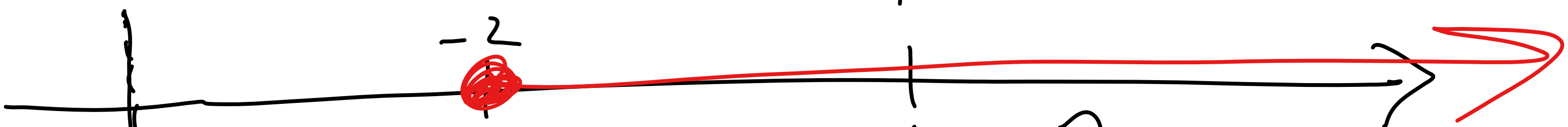


Example 2.4.9. $(x-1)^2(x+2)^3 \geq 0$ positive

$$(x-1)^2(x+2)^3 = 0$$

$x - 1 = 0$ or $x + 2 = 0$

$x = 1$ $\qquad$ $x = -2$



x-value	-5	0	2
$(x-1)^2(x+2)^3$	$(+)(-)$	$(+)(+)$	$(+)(+)$
Conclusion	$-$	$+$	$+$

Answer: $x \geq -2$ $\qquad$ $[-2, \infty)$