2.1 Linear Equations in Two Variables

The simplest mathematical model is the **linear equation in two variables**. The standard form is (**slope-intercept**)

$$\boxed{y = mx + b}$$

where m is the slope and b is the y-intercept. You will recall that

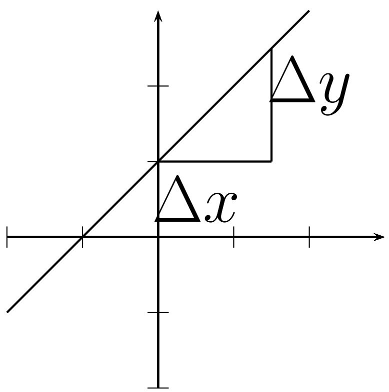
$$\text{slope} = \frac{\text{rise}}{\text{run}}.$$

The slope is the amount of vertical change relative to the horizontal change. Sometimes we think of it as the "change in y" over "change in x".

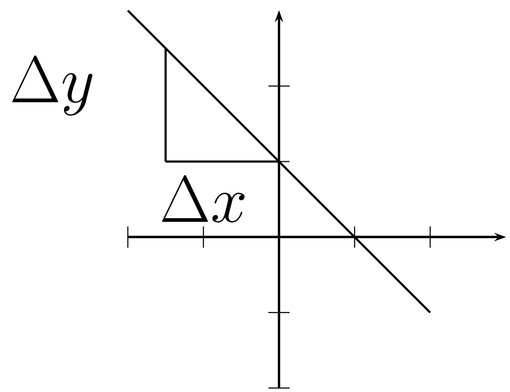
To calculate the slope between two points (x_1, y_1) and (x_2, y_2) the formula is:

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{\Delta y}{\Delta x} = \frac{\text{rise}}{\text{run}}.$$

Positive Slope

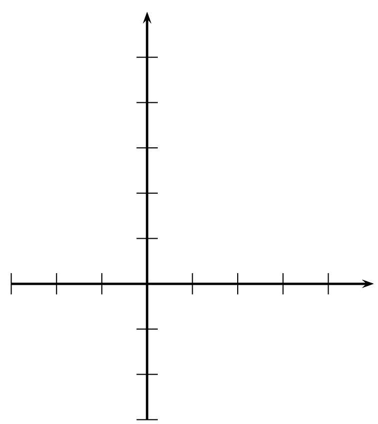


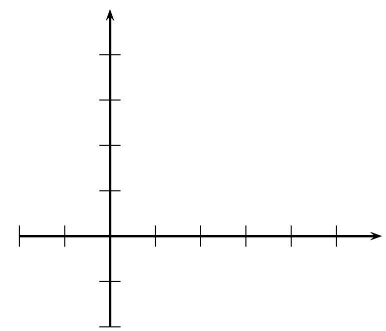
Negative Slope



Example 2.1.1. Sketch the graphs of the following two functions:

$$y = 2x + 3 \qquad\qquad y = -\frac{1}{2}x + 3.$$





Example 2.1.2. Find the slope between the following pairs of points.

(a). (-3, 0) and (4, 4)

(b). (-3, 1) and (4, 1)

(c). (-3, 1) and (-3,4)

2.1.1 Point-Slope Form

$$\boxed{y - y_1 = m(x - x_1)}$$

You always need **two** things:

1. a point: (x_1, y_1) AND
2. a slope m.

Example 2.1.3. Write the equation of the line through (-3,0) and (4, -4). Write the equation in the point slope form and the slope-intecept form.

Example 2.1.4. Write the equation of the lines through

(a). (-3, 1) and (4, 1)

(b). (-3, 1) and (-3,4)

2.1.2 Parallel and Perpendicular Lines

Parallel lines have the same slope.

If $y = m_1x + b_1$ is parallel to $y = m_2x + b_2$ then $m_1 = m_2$.

Perpendicular lines have negative reciprocal slopes.

If $y = m_1x + b_1$ is perpendicular to $y = m_2x + b_2$ then $m_1 = -\frac{1}{m_2}$.

Example 2.1.5. Write the equations of the lines parallel and perpendicular to $-4x + 2y = 3$ passing through the point (2, 1).

2.2 Absolute Value Functions

Definition 2.1. The absolute value of a real number x, denoted $|x|$

$$|x| = \begin{cases} -x, & \text{if } x < 0 \\ x, & \text{if } x \geq 0 \end{cases}$$

Absolute Value Properties

1. **Product rule:** $|ab| = |a||b|$
2. **Power rule:** $|a^n| = |a|^n$
3. **Quotient rule:** $\left|\frac{a}{b}\right| = \frac{|a|}{|b|}$
4. **Equality property 1:** $|x| = 0$ if and only if $x = 0$
5. **Equality property 2:** For $c > 0$, $|x| = c$ if and only if $x = c$ or $x = -c$.
6. **Equality property 1:** For $c > 0$, $|x| = c$ has no solution.

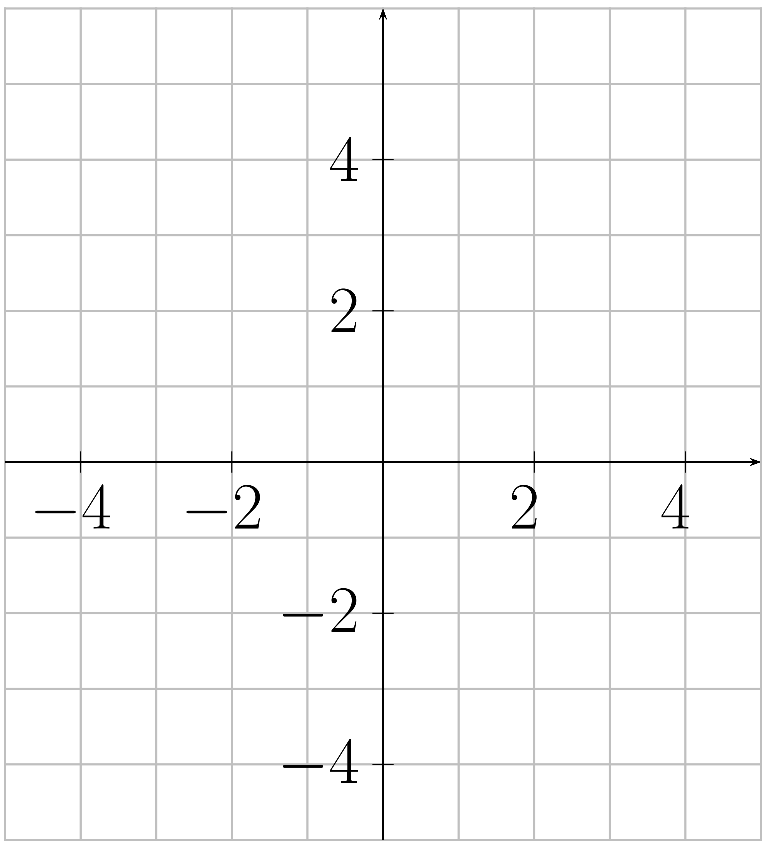
An equation with an absolute value is always TWO equations:

$$\mid x - 5 \mid = 4 \qquad \Longrightarrow \qquad x - 5 = 4 \text{ OR } -(x-5) = 4$$

Example 2.2.1. $-4 + 2 \mid 4x - 4 \mid = 10$

We start by getting the absolute value by itself on one side of the equation.

Example 2.2.2. Sketch a graph of $f(x) = \frac{1}{2}|x - 1| - 3$



2.3 Quadratic Functions

Definition 2.2. Let a, b, and c be real numbers with $a \neq 0$. The function

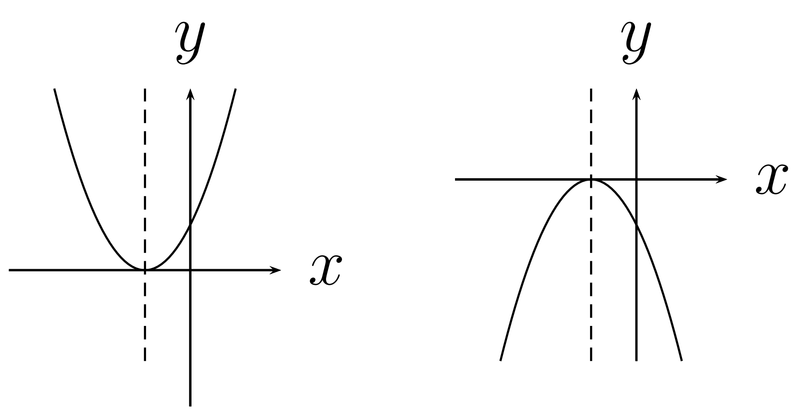
$$f(x) = ax^2 + bx + c$$

is called a **quadratic equation**.

The graph of a quadratic equation is a parabola. All parabolas are symetric with respect to the axis of symetry which passes through the vertex.

Example 2.3.1.

$f(x) = x^2 + 2x + 1$ OR $f(x) = -x^2 - 2x - 1$



$a > 0$ graph opens up

$a < 0$ graph opens down

The Standard Form of a Parabola

$$f(x) = a(x-h)^2 + k$$

Vertex is located at (h, k)

if $a > 0$ graph opens up

if $a < 0$ graph opens down

Alternate form

If

$$f(x) = ax^2 + bx + c$$

then the

$$\text{vertex} = \left(\frac{-b}{2a},\ f\left(\frac{-b}{2a}\right)\right).$$

To Graph a Parabola

1. Find the vertex
2. Find the x - intercepts
3. Determine if it opens up or down
4. Sketch

Example 2.3.2. Graph $f(x) = 2x^2 - 12x - 14$.

Step 1: Vertex:

$x = \frac{-(-12)}{2(2)} = 3$

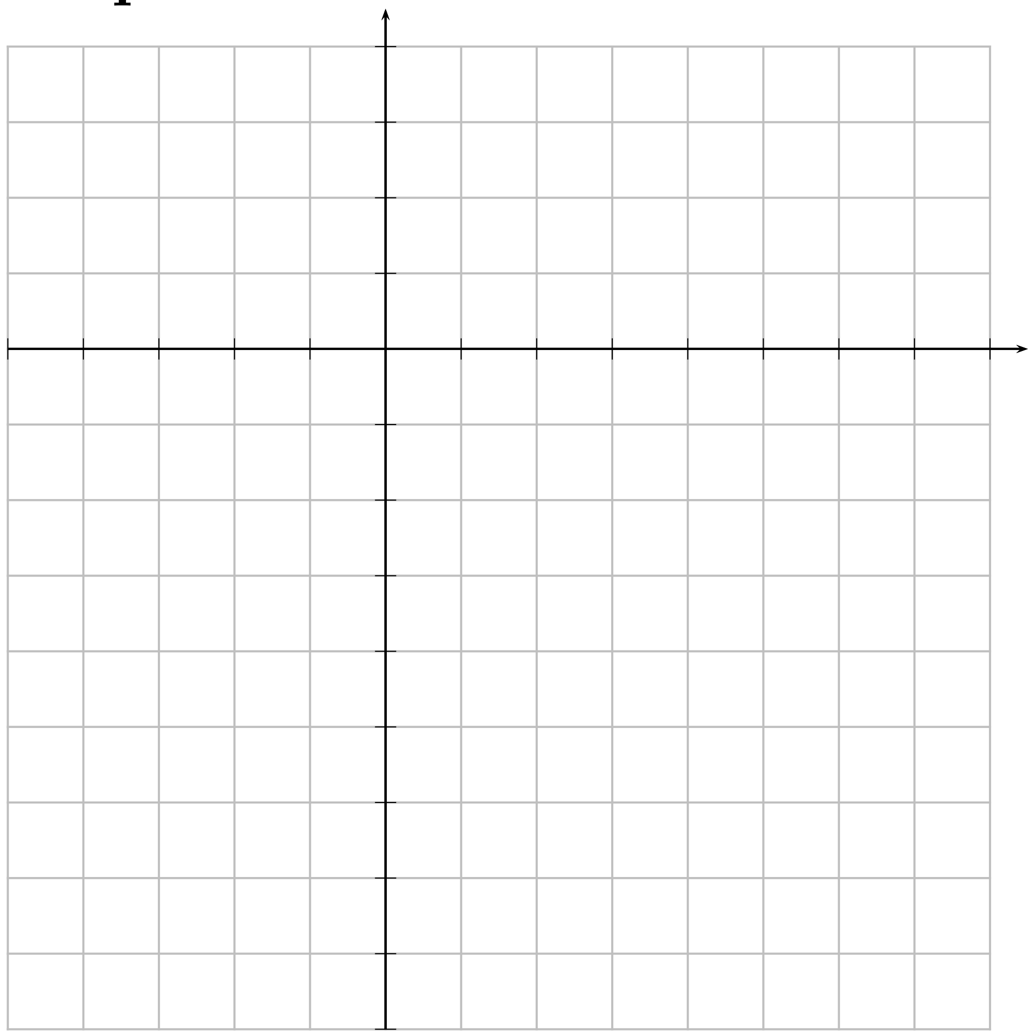
$y = f(3) = 2(9) - 12(3) - 14 = -32$

So the coordinates of the vertex are $(x, y) = (3, -32)$

Step 2: x - intecepts:

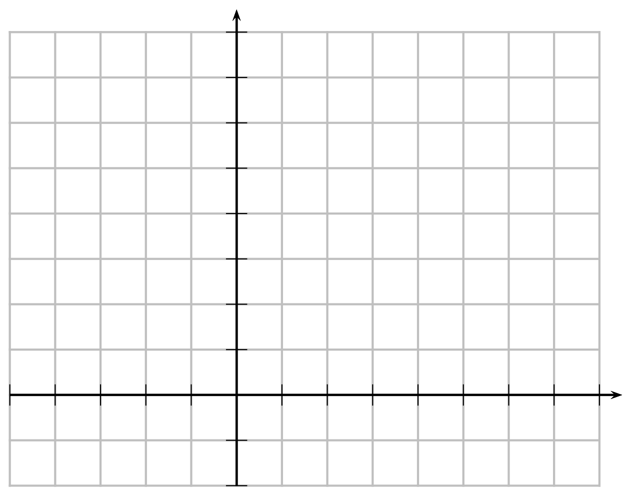
$$0 = 2x^2 - 12x - 14$$

Step 4: Sketch



Example 2.3.3. Graph $f(x) = (x-6)^2 + 3$

Step 4: Sketch



Example 2.3.4. Find the standard form for a parabola that has (0,1) as its vertex and passes through the point (1,0)

Example 2.3.5. Convert to standard form $f(x) = x^2 + 6x + 5$ and graph.

Example 2.3.6. Convert to standard form $f(x) = x^2 - 2x - 8$ and graph.

Example 2.3.7. A rancher has 260 yards of fence with which to enclose three sides of a rectangular meadow (the fourth side is a river and will not require fencing). Find the dimensions of the meadow with the largest possible area.

Example 2.3.8. A person standing close to the edge on top of a 72-foot building throws a ball vertically upward. The quadratic function $h(t) = -16t^2 + 84t + 72$ models the ball's height about the ground, $h(t)$, in feet, t seconds after it was thrown.

1. What is the maximum height of the ball?

2. How many seconds does it take until the ball hits the ground?

2.4 Solving Inequalities with Absolute Value and Quadratic Functions

2.4.1 Graphing inequalities

Properties of Inequalities

1. Transitive: $a < b$ and $b < c \implies a < c$
2. Addition of Constants: If $a < b$ then $a + c < b + c$
3. Addition: If $a < b$ and $c < d$ then $a + c < b + d$
4. Multiplication by a constant:

   If $c > 0$ and $a < b$ then $ac < bc$

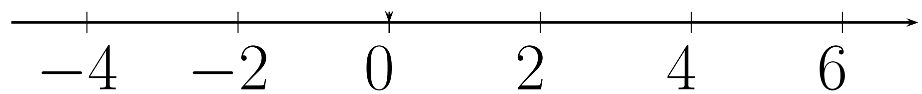
   If $c < 0$ and $a < b$ then $ac > bc$

NOTE: If you multiply or divide by a negative number you reverse the order of the inequality.

2.4.2 Solving linear inequalities

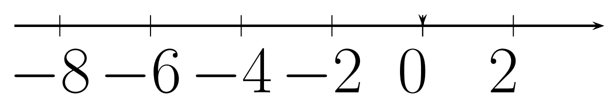
Example 2.4.1. $10x < 40$

Example 2.4.2. $-10x < 40$



Example 2.4.3. $4(x+1) \leq 2x+3$

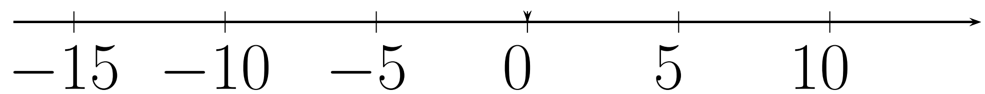
Example 2.4.4. $-8 \leq -(3x+5) < 13$



2.4.3 Absolute value and inequalities

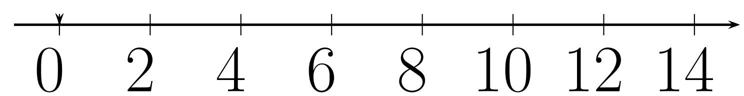
Absolute value is still two equations

Example 2.4.5. $\left| \frac{x}{2} \right| > 5$



$$\frac{x}{2} > 5 \text{ OR } -\frac{x}{2} > 5$$

Example 2.4.6. $| x - 7 | < 5$



$$-5 < x - 7 < 5$$

Question: What does $| x - 2 | < 5$ mean?
Answer: All real numbers within five units of two.

So all real numbers within 5 units of 8 would be written as:

And all real numbers at least 5 units from 8 would be written as:

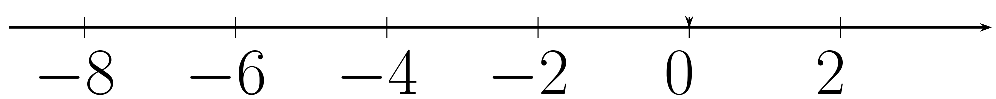
2.4.4 Solving polynomial inequalities

Example 2.4.7. $x^2 < 5$

Step 1: Set equation equal to zero and find the zeros.

Step 2: Set up a table of signs

Step 3: Find where the table gives negative values and write the solution



Example 2.4.8. $x^2 + 2x - 3 \geq 0$

Example 2.4.9. $(x-1)^2(x+2)^3 \geq 0$